Technical Appendix: On the Size of the Fiscal Multiplier When the Nominal Interest Rate is Zero

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This technical appendix is organized as follows. In Section 1, we describe the equilibrium conditions for two specifications of price adjustment, Calvo and Rotemberg. In Section 2 we go through the derivation of the equilibrium condition for the Calvo specification, following Shmitt-Grohe and Uribe (2007).

1 Equilibrium condition

1.1 Calvo Specification

$$
\begin{aligned}
0 &= \beta d_{t+1} \frac{u_{c,t+1}}{u_{c,t}} \frac{1+R_{t+1}}{1+\pi_{t+1}} - 1 && (1) \\
0 &= \frac{u_{l,t}}{u_{c,t}} - w_t && (2) \\
0 &= \beta d_{t+1} \frac{u_{c,t+1}}{u_{c,t}} \Big[r_{t+1} + q_{t+1} \big\{ 1 - \delta - \frac{\sigma_I}{2} (\frac{x_{t+1}}{k_{t+1}} - \delta)^2 + \sigma_I (\frac{x_{t+1}}{k_{t+1}} - \delta) \frac{x_{t+1}}{k_{t+1}} \big\} \Big] - q_t && (3) \\
0 &= q_t [1 - \sigma_I (\frac{x_t}{k_t} - \delta)] - 1 && (4) \\
0 &= R_{t+1} - \big\{ (1 + R_{ss}) (\frac{1+\pi_t}{1+\pi_{ss}})^{\rho_p} - 1 \big\}, \quad \forall t \leq S - 1 \& t \geq T + 1 && (5) \\
0 &= R_{t+1}, \quad \forall t \in \{S, ... T\}. && (6) \\
0 &= r_t - \frac{\alpha}{1-\alpha} w_t \frac{h_t}{k_t} && (7) \\
0 &= mc_t - \frac{r_t^{\alpha} w_t^{1-\alpha}}{\alpha^{\alpha} (1-\alpha)^{1-\alpha} A_t^{1-\alpha}} && (8) \\
0 &= x_t + (1-\delta) k_t - \frac{\sigma_I}{2} (\frac{x_t}{k_t} - \delta)^2 k_t - k_{t+1} && (9) \\
0 &= c_t + x_t + g_t - y_t && (10) \\
0 &= y_t^g - k_t^{\alpha} \big(A_t L_t \big)^{1-\alpha} && (11) \\
0 &= y_t - \frac{1}{PD_t^{-\theta}} y_t^g && (12) \\
0 &= (1-\theta) as_{1,t} + \theta (1 - \tau_{mc}) as_{2,t} && (13) \\
0 &= -as_{1,t} + \tilde{p}_t^{-\theta} y_t + \beta \gamma d_{t+1} \frac{u_{c,t+1}}{u_{c,t}} \big(\frac{1}{1+\pi_{t+1}} \big)^{1-\theta} \big(\frac{\tilde{p}_t}{\tilde{p}_{t+1}} \big)^{-\theta} as_{1,t+1} && (14) \\
0 &= -as_{2,t} + mc_t \tilde{p}_t^{-\theta-1} y_t + \beta \gamma d_{t+1} \frac{u_{c,t+1}}{u_{c,t}} \big(\frac{1}{1+\pi_{t+1}} \big)^{-\theta} \big(\frac{\tilde{p}_t}{\tilde{p}_{t+1}} \big)^{-\theta-1} as_{2,t+1} && (15) \\
0 &= -PD_t^{-\theta} + (1-\gamma) \tilde{p}_t^{-\theta} + \gamma \big(\frac{1}{1+\pi_t} \big)^{-\theta} PD_{t-1}^{-\theta} && (16) \\
0 &= -1 + (1-\gamma) \tilde{p}_t^{1-\theta} + \gamma \big(\frac{1}{1+\pi_t} \big)^{1-\theta} && (17)
\end{aligned}
$$

Equations (1)-(4) are household's first order condition. Equations (5) and (6) are Taylor rule.

Firm's cost minimization yields equations (7) and (8). Equations (9)-(12) are capital accumulation equation and resource constraint. The rest are obtained from optimality of price setting behavior, which we go through in detail in Section 2.

1.2 Rotemberg Specification

$$0 = \beta d_{t+1}\frac{u_{c,t+1}}{u_{c,t}}\frac{1+R_{t+1}}{1+\pi_{t+1}} - 1 \tag{18}$$

$$0 = \frac{u_{l,t}}{u_{c,t}} - w_t \tag{19}$$

$$0 = \beta d_{t+1}\frac{u_{c,t+1}}{u_{c,t}}\Big[r_{t+1} + q_{t+1}\big\{1-\delta-\frac{\sigma_I}{2}(\frac{x_{t+1}}{k_{t+1}}-\delta)^2 + \sigma_I(\frac{x_{t+1}}{k_{t+1}}-\delta)\frac{x_{t+1}}{k_{t+1}}\big\}\Big] - q_t \tag{20}$$

$$0 = q_t[1-\sigma_I(\frac{x_t}{k_t}-\delta)] - 1 \tag{21}$$

$$0 = R_{t+1} - \big\{(1+R_{ss})(\frac{1+\pi_t}{1+\pi_{ss}})^{\rho_p} - 1\big\}, \quad \forall t \leq S-1 \& t \geq T+1 \tag{22}$$

$$0 = R_{t+1}, \quad \forall t \in \{S, ...T\}. \tag{23}$$

$$0 = r_t - \frac{\alpha}{1-\alpha}w_t\frac{h_t}{k_t} \tag{24}$$

$$0 = mc_t - \frac{r_t^{\alpha}w_t^{1-\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}A_t^{1-\alpha}} \tag{25}$$

$$0 = x_t + (1-\delta)k_t - \frac{\sigma_I}{2}(\frac{x_t}{k_t}-\delta)^2 k_t - k_{t+1} \tag{26}$$

$$0 = c_t + x_t + g_t - y_t \tag{27}$$

$$0 = y_t^g - k_t^{\alpha}(A_t h_t)^{1-\alpha} \tag{28}$$

$$0 = y_t - \{1-\Gamma(\frac{1+\pi_t}{1+\pi_{ss}})\}y_t^g \tag{29}$$

$$0 = 1-\theta+\theta(1-\tau_{mc})mc_t - \Gamma'(\frac{1+\pi_t}{1+\pi_{ss}})\frac{1+\pi_t}{1+\pi_{ss}} + \beta d_{t+1}\frac{u_{c,t+1}}{u_{c,t}}\frac{y_{t+1}^g}{y_t^g}\Gamma'(\frac{1+\pi_{t+1}}{1+\pi_{ss}})\frac{1+\pi_{t+1}}{1+\pi_{ss}} \tag{30}$$

Equations (18)-(28) are the same as those in Calvo specification. Equations (29) is the resource constraint that reflects the price adjustment costs. Equation (20) is obtained from optimality of price setting behavior and the assumed symmetry across intermediate firms.

1.3 Initial and Terminal Conditions

In Rotemberg specification, capital is only one state variable, and its initial condition is set to the steady state value: $k_0 = k_{ss}$. We truncate the economy at period T^* and assume/impose that the economy is in the steady state from $T^* + 1$ on. This gives the terminal condition: $(c_{T^*+1}, k_{T^*+2}, y_{T^*+1}, r_{T^*+1}, \pi_{T^*+1}) = (c_{ss}, k_{ss}, y_{ss}, r_{ss}, \pi_{ss})$.

In Calvo specification one needs to choose the initial/terminal conditions for additional state variables. Variables as_t's are forward-looking auxiliary state variables, which we discuss in detail in Section 2. We set its terminal condition and the initial values are free. While PD_{-1} has to be specified in a rather arbitrary way. We set $PD_{-1} = 1$, i.e. no initial price dispersion.

We have the same number of equations and unknowns, thus for a given pair of (S, T) and a given sequence of exogenous variables, we can solve for a sequence of prices and quantities.

2 Details on the derivation of equilibrium condition for Calvo specification.

The first order condition associated with the choice of $\pi_t(i)$ is:

$$0 = \sum_{j=0}^{\infty} \Lambda_{t,t+j} \frac{P_t}{P_{t+j}} \gamma^j \Big\{ (1-\theta) \big(\frac{p_t(i)}{P_{t+j}} \big)^{-\theta} y_{t+j} + (1-\tau_\chi) \theta P_{t+j} mc_{t+j} \big(\frac{p_t(i)}{P_{t+j}} \big)^{-\theta-1} y_{t+j} \Big\}. \tag{31}$$

where $\Lambda_{t,t+j}$ is the discount factor between periods t and $t+j$ that in equilibrium equals to the representative household's: $\beta^j \big(\prod_{s=1}^{j} d_{t+s} \big) \frac{u_{c,t+j}}{u_{c,t}}$. Under the assumption that households accumulate capital and firms rent capital in competitive markets, a typical firm i that can adjust its price i in period t will choose $\tilde{P}_t$ to satisfy

$$0 = \sum_{j=0}^{\infty} \Lambda_{t,t+j} \frac{P_t}{P_{t+j}} \gamma^j \Big\{ (1-\theta) \big(\frac{\tilde{P}_t}{P_{t+j}} \big)^{-\theta} y_{t+j} + (1-\tau_\chi) \theta P_{t+j} mc_{t+j} \big(\frac{\tilde{P}_t}{P_{t+j}} \big)^{-\theta-1} y_{t+j} \Big\}. \tag{32}$$

This equation is nonlinear in $p_t(i)$ and forward-looking. When solving the model exactly we will need to keep track of how firms who are allowed to adjust prices set their price in all periods. We follow Schmitt-Grohe and Uribe (2006) and introduce two auxiliary state variables:

$$as_{1,t} = \sum_{j=0}^{\infty} \Lambda_{t,t+j} \frac{P_t}{P_{t+j}} \gamma^j \Big\{ (\frac{\tilde{P}_t}{P_{t+j}})^{-\theta} y_{t+j} \Big\}. \tag{33}$$

$$as_{2,t} = \sum_{j=0}^{\infty} \Lambda_{t,t+j} \frac{P_t}{P_{t+j}} \gamma^j \Big\{ P_{t+j} mc_{t+j} (\frac{\tilde{P}_t}{P_{t+j}})^{-\theta-1} y_{t+j} \Big\}. \tag{34}$$

These state variables follow the recursion formulas:

$$\begin{aligned}
as_{1,t} &= (\frac{\tilde{P}_t}{P_t})^{-\theta} y_t + \gamma \Lambda_{t,t+1} \frac{P_t}{P_{t+1}} \sum_{j=0}^{\infty} \Lambda_{t+1,t+1+j} \frac{1}{P_{t+1+j}} \gamma^j \Big\{ (\frac{\tilde{P}_t}{P_{t+1+j}})^{-\theta} y_{t+1+j} \Big\}. \\
&= (\frac{\tilde{P}_t}{P_t})^{-\theta} y_t + \gamma \Lambda_{t,t+1} \frac{P_t}{P_{t+1}} (\frac{\tilde{P}_t}{\tilde{P}_{t+1}})^{-\theta} \sum_{j=0}^{\infty} \Lambda_{t+1,t+1+j} \frac{1}{P_{t+1+j}} \gamma^j \Big\{ (\frac{\tilde{P}_{t+1}}{P_{t+1+j}})^{-\theta} y_{t+1+j} \Big\}. \\
&= (\frac{\tilde{P}_t}{P_t})^{-\theta} y_t + \beta d_{t+1} \gamma \frac{u_{c,t+1}}{u_{c,t}} \frac{P_t}{P_{t+1}} (\frac{\tilde{P}_t}{\tilde{P}_{t+1}})^{-\theta} as_{1,t+1}.
\end{aligned}$$

$$\begin{aligned}
as_{2,t} &= \sum_{j=0}^{\infty} \Lambda_{t,t+j} \frac{P_t}{P_{t+j}} \gamma^j \Big\{ P_{t+j} mc_{t+j} (\frac{\tilde{P}_t}{P_{t+j}})^{-\theta-1} y_{t+j} \Big\}. \\
&= mc_t (\frac{\tilde{P}_t}{P_t})^{-\theta-1} y_t + \gamma \Lambda_{t,t+1} \frac{P_t}{P_{t+1}} \sum_{j=0}^{\infty} \Lambda_{t+1,t+1+j} \frac{P_{t+1}}{P_{t+1+j}} \gamma^j \Big\{ P_{t+1+j} mc_{t+1+j} (\frac{\tilde{P}_t}{P_{t+1+j}})^{-\theta-1} y_{t+1+j} \Big\}. \\
&= mc_t (\frac{\tilde{P}_t}{P_t})^{-\theta-1} y_t + \gamma \Lambda_{t,t+1} \frac{P_t}{P_{t+1}} (\frac{\tilde{P}_t}{\tilde{P}_{t+1}})^{-\theta-1} \sum_{j=0}^{\infty} \Lambda_{t+1,t+1+j} \frac{P_{t+1}}{P_{t+1+j}} \gamma^j \Big\{ P_{t+1+j} mc_{t+1+j} (\frac{\tilde{P}_{t+1}}{P_{t+1+j}})^{-\theta-1} y_{t+1+j} \Big\} \\
&= mc_t (\frac{\tilde{P}_t}{P_t})^{-\theta-1} y_t + \beta d_{t+1} \gamma \frac{u_{c,t+1}}{u_{c,t}} \frac{P_t}{P_{t+1}} (\frac{\tilde{P}_t}{\tilde{P}_{t+1}})^{-\theta-1} as_{2,t+1}.
\end{aligned}$$

Since nominal prices can grow over time, we detrend them by deflating them by P_t. Define

$\tilde{p}_t = \tilde{P}_t / P_t$. Then we have

$$\begin{aligned} as_{1,t} &= \tilde{p}_t^{-\theta} y_t + \beta d_{t+1} \gamma \frac{u_{c,t+1}}{u_{c,t}} (\frac{P_t}{P_{t+1}})^{1-\theta} (\frac{\tilde{p}_t}{\tilde{p}_{t+1}})^{-\theta} as_{1,t+1} \\ &= \tilde{p}_t^{-\theta} y_t + \beta d_{t+1} \gamma \frac{u_{c,t+1}}{u_{c,t}} (\frac{1}{1+\pi_{t+1}})^{1-\theta} (\frac{\tilde{p}_t}{\tilde{p}_{t+1}})^{-\theta} as_{1,t+1}, \end{aligned}$$

and

$$\begin{aligned} as_{2,t} &= mc_t \tilde{p}_t^{-\theta-1} y_t + \beta d_{t+1} \gamma \frac{u_{c,t+1}}{u_{c,t}} (\frac{P_t}{P_{t+1}})^{-\theta} (\frac{\tilde{p}_t}{\tilde{p}_{t+1}})^{-\theta-1} as_{2,t+1} \\ &= mc_t \tilde{p}_t^{-\theta-1} y_t + \beta d_{t+1} \gamma \frac{u_{c,t+1}}{u_{c,t}} (\frac{1}{1+\pi_{t+1}})^{-\theta} (\frac{\tilde{p}_t}{\tilde{p}_{t+1}})^{-\theta-1} as_{2,t+1}. \end{aligned}$$

Using these two state variables the first order condition for setting prices can be expressed as

$$(1-\theta) as_{1,t} + (1-\tau_{mc}) \theta as_{2,t} = 0. \tag{35}$$

Next recall that the price level aggregator is:

$$P_t = [\int_0^1 p_t(i)^{1-\theta} di]^{\frac{1}{1-\theta}}. \tag{36}$$

Then for all t we have:

$$\begin{aligned} 1 &= \int_0^1 (\frac{p_t(i)}{P_t})^{1-\theta} di \\ &= \sum_{j=0}^{\infty} (1-\gamma)\gamma^j (\frac{\tilde{P}_{t-j}}{P_t})^{1-\theta} \\ &= (1-\gamma)\tilde{p}_t^{1-\theta} + \gamma (\frac{P_{t-1}}{P_t})^{1-\theta} \sum_{j=0}^{\infty} (1-\gamma)\gamma^j (\frac{\tilde{P}_{t-1-j}}{P_{t-1}})^{1-\theta} \\ &= (1-\gamma)\tilde{p}_t^{1-\theta} + \gamma (\frac{1}{1+\pi_t})^{1-\theta}. \end{aligned}$$

We will see below that one of the key distinctions between the loglinearized version of the model and the exact nonlinear version of the model is that the procedure of loglinearizing around a steady-state with zero inflation sets the resource costs of price dispersion to zero. Here we derive the resource costs of price dispersion. Recall that

$$y_t(i) = \big(p_t(i)/P_t\big)^{-\theta} y_t \tag{37}$$

and

$$y_t(i) = k_t(i)^{\alpha} \big(A_t L_t(i)\big)^{1-\alpha}. \tag{38}$$

Thus

$$\begin{aligned}
\int_0^1 y_t(i) di &= \int_0^1 k_t(i)^{\alpha} \big(A_t L_t(i)\big)^{1-\alpha} di \\
&= \int_0^1 (1/\alpha) r_t k_t(i) di \\
&= (1/\alpha) r_t \int_0^1 k_t(i) di \\
&= k_t^{\alpha} \big(A_t L_t\big)^{1-\alpha}
\end{aligned}$$

and

$$\begin{aligned}
\int_0^1 y_t(i) di &= \int_0^1 \big(p_t(i)/P_t\big)^{-\theta} y_t di \\
&= \big[\int_0^1 \big(p_t(i)/P_t\big)^{-\theta} di\big] y_t.
\end{aligned}$$

Then the resource constraint for this economy is

$$c_t + x_t + g_t = y_t = \big[\int_0^1 \big(p_t(i)/P_t\big)^{-\theta} di\big]^{-1} k_t^{\alpha} \big(A_t L_t\big)^{1-\alpha}. \tag{39}$$

Define $PD_t = [\int_0^1 \left(p_t(i)/P_t\right)^{-\theta} di]^{-1/\theta}$ to be the price distortion. Then

$$\begin{aligned}
PD_t^{-\theta} &= \int_0^1 \left(p_t(i)/P_t\right)^{-\theta} di \\
&= \sum_{j=0}^{\infty}(1-\gamma)\gamma^j \left(\frac{\tilde{P}_{t-j}}{P_t}\right)^{-\theta} \\
&= (1-\gamma)\tilde{p}_t^{-\theta} + \gamma\left(\frac{P_{t-1}}{P_t}\right)^{-\theta} PD_{t-1} \\
&= (1-\gamma)\tilde{p}_t^{-\theta} + \gamma\left(\frac{1}{1+\pi_t}\right)^{-\theta} PD_{t-1}^{-\theta}
\end{aligned}$$

and the resource constraint becomes

$$c_t + x_t + g_t = y_t = \frac{1}{PD_t^{-\theta}} k_t^{\alpha}\left(A_t L_t\right)^{1-\alpha}. \tag{40}$$

References

[1] Schmitt-Grohe, Stephanie, and Martin Uribe (2007), “Optimal Simple and Implementable Monetary and Fiscal Rules,” *Journal of Monetary Economics* 54, 1702-1725.