Online Appendix: Some unpleasant properties of loglinearized solutions when the nominal rate is zero.[*]

R. Anton Braun[†] Lena Boneva[‡] Yuichiro Waki[§]

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[*]These views are our own and not those of the Federal Reserve System or the Bank of England.

[†]Federal Reserve Bank of Atlanta; email: r.anton.braun@gmail.com; address: 1000 Peachtree St NE, Atlanta, GA 30309, USA

[‡]Bank of England; email: lena.boneva@bankofengland.co.uk; address: Threadneedle St, London EC2R 8AH, UK

[§]University of Queensland; email: y.waki@uq.edu.au; address: St Lucia QLD 4072, Australia

Appendix A Alternative parameterizations of the model

This section provides additional information about the properties of the model using alternative configurations of the model's parameters. We vary the risk aversion coefficient σ and report results using a calibration strategy that holds the level of technology fixed. We also consider parameterizations of the model based on Christiano and Eichenbaum (2012), Denes, Eggertsson, and Gilbukh (2013) and Eggertsson (2011).

A.1 Baseline calibration strategy using alternative values of σ

Figure 1 reports the regions where each of the four cases of equilibrium occur using the NL and LL equilibrium conditions for $p \in [0, 0.985]$ and $\sigma \in [0.25, 2]$. The other parameters are held fixed at their baseline values and the shocks $\{d^L, z^L\}$ are adjusted to reproduce the GR inflation and GDP targets.

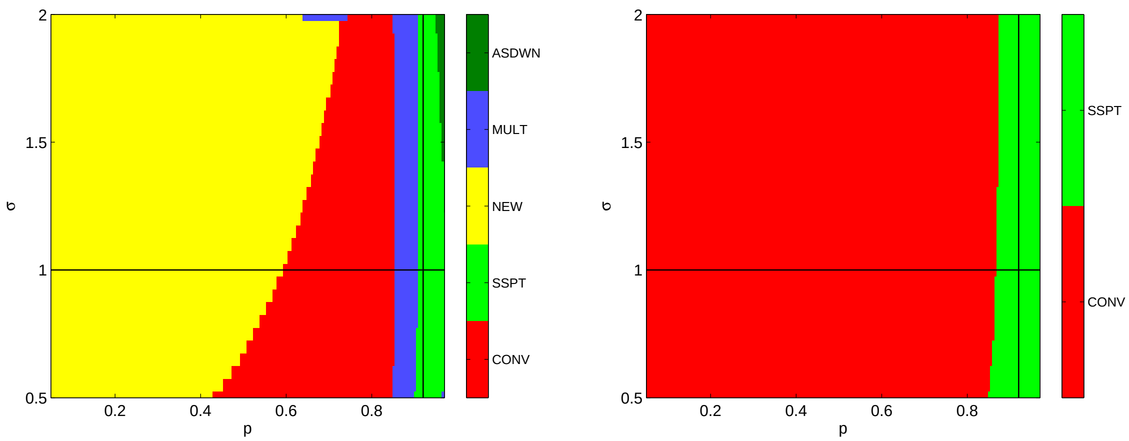
We know from equations (13) – (14) in the paper that a higher value of σ increases both $slope(AS)$ and $slope(AD)$. Figure 1 shows that these changes in turn act to increase the size of the region where the New ZLB equilibrium occurs and reduce the size of the region where the Conventional ZLB equilibrium occurs. The size of the regions where either Sunspot ZLB equilibria or Multiple ZLB equilibria occur do not change much. However, there is now a small Multiple ZLB region to the left of the asymptote when $\sigma = 2$.[1] It occurs in the region where $slope(AD)$ changes sign. For most of this region the targeted equilibrium satisfies $slope(AD) < 0 < slope(AS)$ and the nontargeted equilibrium satisfies $slope(AS) < slope(AD) < 0$. Using our baseline value of γ and $\sigma = 2$ a ZLB equilibrium with $slope(AD) < 0 < slope(AS)$ occurs when $p \leq 0.725$.

Note also that there is a new type of ZLB equilibrium. It occurs for high values of σ and large $p \geq 0.95$. This ZLB equilibrium has a downward sloping AS schedule: $slope(AS) < 0 < slope(AD)$. This equilibrium which, is a variant of a sunspot equilibrium because there is a second equilibrium with a positive interest rate, only arises using the NL equilibrium conditions.

A larger value of σ results in small and orthodox fiscal multipliers for a larger range of p's as compared to our baseline parameterization. (Compare e.g. $\sigma = 2$ in 2a and 2b with $\sigma = 1$). This result is an immediate implication of the fact a larger σ increases the range of p's where a New equilibrium occurs and shrinks the size of the region with a Conventional equilibrium.

[1] This situation is rare in the sense that it only occurs when σ takes on an integer value of 2 and p is between 0.64 and 0.74. For other values of σ in this neighborhood, all but one of the roots of the system are complex and the equilibrium is globally unique.

Figure 1: Types of ZLB Equilibria For Alternative Values of Risk Aversion: Baseline Calibration



(a) NL Equilibrium

(b) LL Equilibrium

Notes: CONV: Conventional ZLB equilibrium (slope(AD)>slope(AS)>0); SSPT: Sunspot ZLB equilibrium (slope(AS)>slope(AD)>0); NEW: New ZLB equilibrium (slope(AD)<0<slope(AS)); MULT: Multiple ZLB equilibria; ASDWN: ZLB equilibrium with downward sloping AS schedule (slope(AS)<0<slope(AD)).

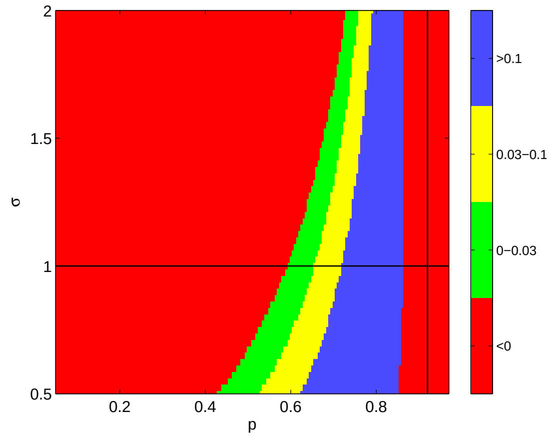
A larger value of σ thus increases the range of values of p where the LL solution misclassifies the type of equilibrium and incorrectly infers that the sign of the labor tax fiscal multiplier is positive. This result can be readily discerned by comparing Figures 2a and 2c. However, the bias in the government purchase multiplier is not affected much by the choice of σ (compare Figures 2b and 2d). The LL solution correctly infers that the region with small government purchase multipliers is larger when σ is increased.

A.2 No-technology-shock calibration scheme

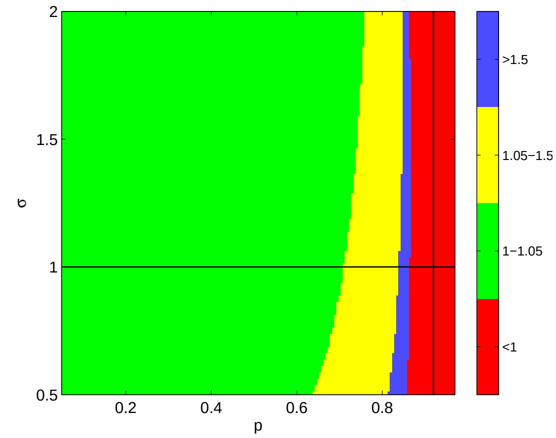
Most analyses of the ZLB that use the Eggertsson and Woodford (2003) Markov equilibrium posit a single shock to demand. We next present results that use a single shock.

Each parameterization of the model we have displayed up to this point reproduces the GR declines in output and inflation. If we are to continue to reproduce these two facts with $z^L = z = 1$, we need to adjust another parameter instead. We choose to adjust the Dixit-Stiglitz parameter, θ. Using this alternative calibration strategy it turns out that θ adjusts with $p\beta d^L$ in such a way as to keep $\theta/(1 - p\beta d^L)$ constant in equation (13) which in turn renders $slope(AS)$ independent of p. For our baseline parameterization of the model, this means that $slope(AS)$ will be equal to 0.036 for all choices of p. This is about the same value of $slope(AS)$ that occurs using the baseline calibration scheme with technology shocks

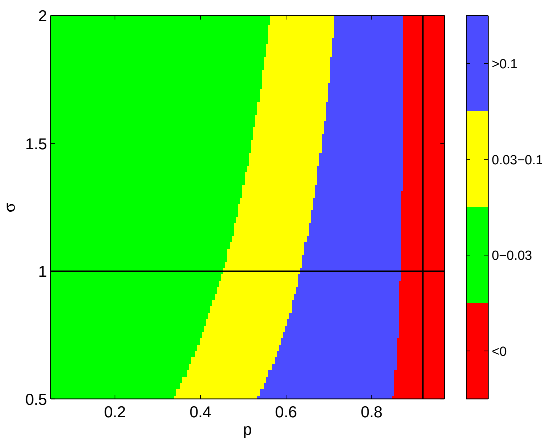
Figure 2: Fiscal multipliers For Alternative Values of p and σ: Baseline calibration



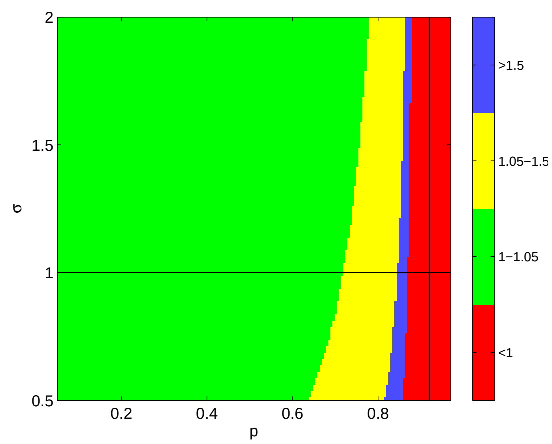
(a) Labor-tax-multiplier, NL solution.



(b) Government-purchase-GDP-multiplier, NL solution.



(c) Labor-tax-multiplier, LL solution.



(d) Government-purchase-GDP-multiplier, LL solution.

Notes: The black line denotes the baseline value of each parameter. Red: Labor tax multiplier is negative (employment increases when the labor tax is cut); Red: the government-purchase-GDP-multiplier is less than 1; Green: Labor tax multiplier is in $[0, 0.03]$; Green: the government-purchase-GDP-multiplier is in $[1, 1.05]$; Yellow: Labor tax multiplier is in $(0.03, 0.1]$; Yellow: the government-purchase-GDP-multiplier is in $[1.05, 1.5]$; Blue: labor tax multiplier exceeds 0.1; Blue: the government-purchase-GDP-multiplier exceeds 1.5.

for $p \approx 0.42$.

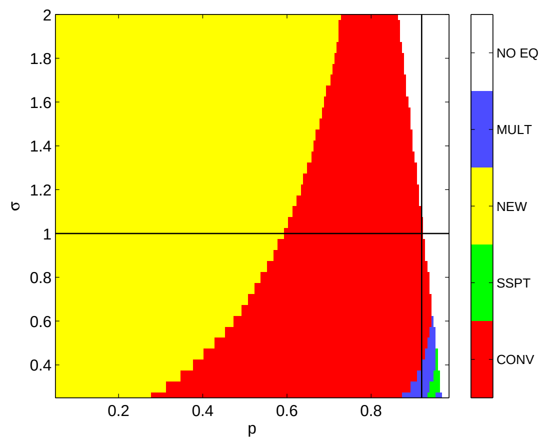
The fact that the AS curve no longer varies with p has two main consequences. The first consequence is that the locus of $p's$ where $slope(AD)$ and $slope(AS)$ become equal and then cross is shifted to the right. This is noteworthy because it results in a larger region of the parameter space with a Conventional ZLB equilibrium. Comparing Figure 3 with Figure 1a and Figure 4a in the paper we see that the region with a Conventional ZLB equilibrium starts when p takes on about the same value using either calibration strategy. However, under the alternative calibration scheme, a Conventional ZLB equilibrium obtains for larger values of p. In particular, our baseline parameterization of the model now produces a Conventional equilibrium. The size of the two indeterminacy regions (Sunspot and Multiple ZLB equilibria) are correspondingly smaller. However, there is still a large region with a New equilibrium. The LL solution misclassifies the type of equilibrium here to be Conventional and thus produces an incorrect sign of the labor tax multiplier. Panels a) and b) of Figure 4 have large red regions to the left of the asymptote where employment increases when the labor tax is cut using the NL solution. In panels c) and d) these regions are green, indicating that employment falls when the labor tax is cut according to the LL solution.

A second consequence of using the alternative calibration scheme is that the AS curve is now flatter at higher values of p and the fiscal multipliers are correspondingly smaller. This effect can be readily discerned for the labor tax multiplier by comparing Figure 7a) in the paper with the upper panel of Figure 4.[2] This difference is even more pronounced for the government purchase multiplier. For instance the yellow region with government purchase multipliers between 1.05 and 1.5 begins at $p = 0.71$ in Figure 7b) in the paper using our baseline parameterization. In Figure 5 the yellow region does not begin until p reaches a value of 0.84. In fact, the government purchase multiplier is less than 1.5 for all choices of $p \leq 0.95$ using the no-technology-shock calibration scheme. It is only in the immediate neighborhood of the asymptote, which occurs at $p \approx 0.965$, that the government purchase multiplier exceeds 1.5. Thus, the government purchase multiplier is less than 1.5 using our preferred value of $p = 0.92$.

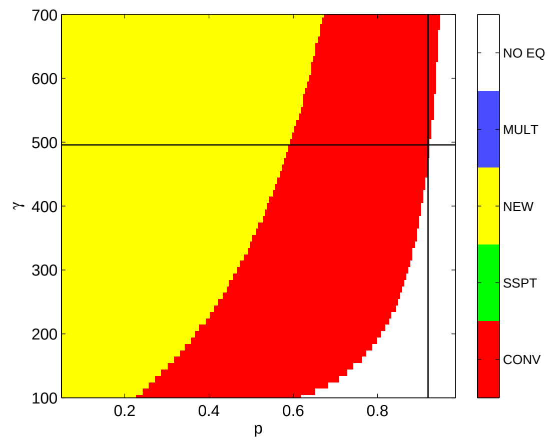
So far we have not said anything about the range of values taken on by θ. Higher values of p are associated with a smaller value of θ, and θ is declining in σ and γ. Some of the results reported in Figures 3–5 need $\theta < 1$ to hit the GR targets. These regions are reported in white. Even though we can compute solutions with values of $\theta < 1$ due to our subsidy scheme, θ in this range imply negative markups and are not of economic interest. To provide more information about when this occurs suppose that γ is held fixed at its baseline value of

[2] In the paper labor tax multipliers exceed 0.1 beginning at $p = 0.72$ for our baseline parameterization. In Figure 4, in contrast, it occurs at $p = 0.81$.

Figure 3: Types of ZLB Equilibria For Alternative Values of Risk Aversion and Price Adjustment Costs: No Technology Shock



(a) Alternative Combinations of p and σ.



(b) Alternative Combinations of p and γ.

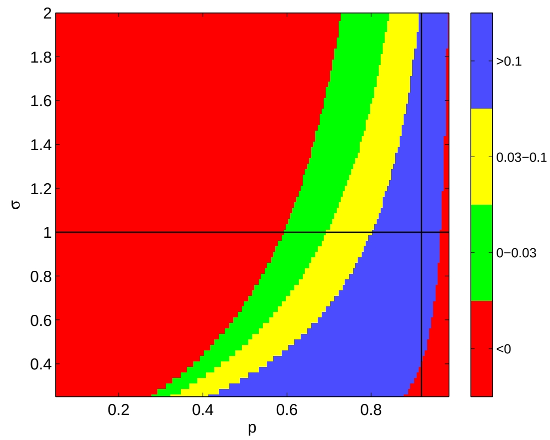
Notes: Red: Conventional ZLB equilibrium (slope(AD)>slope(AS)>0); Green: Sunspot ZLB equilibrium (slope(AS)>slope(AD)>0); Yellow: New ZLB equilibrium (slope(AD)<0<slope(AS)); Blue: Multiple ZLB equilibria; White: $\theta < 1$

495.8 and $\sigma = 1$ then θ falls below 1 when p reaches 0.925. This value of p is only marginally higher than our preferred value of $p = 0.92$. The associated values of the labor tax and government purchase multipliers are 0.58 and 1.15 respectively.

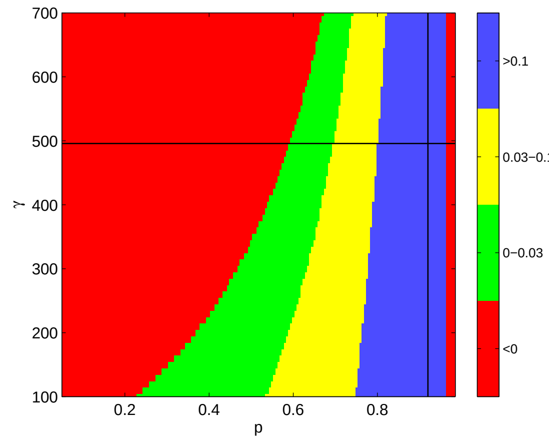
Most empirical estimates of θ are two or higher (see e.g. Broda and Weinstein (2004)). If we use our baseline parameterization of the model and limit attention to values of $\theta \geq 2$, then $p \geq 0.85$ are ruled out. Imposing this restriction implies that the labor tax multiplier is 0.17 or less and that the government purchase GDP multiplier is 1.05 or less. From this we see that our main conclusions continue to obtain when the model is calibrated in this way instead.

Overall, the general pattern of results that emerges using this calibration scheme is consistent with the results reported in the paper. We find large regions where the LL solution produces an incorrect sign for the labor tax multiplier. In addition, the magnitudes of both fiscal multipliers is smaller using this calibration scheme for many settings of p. Perhaps the biggest difference between the two calibration schemes is that the size of the indeterminacy regions is much smaller if the technology shock is held fixed. This follows from the fact that a flatter AS schedule acts to push the asymptote as indexed by p to the right.

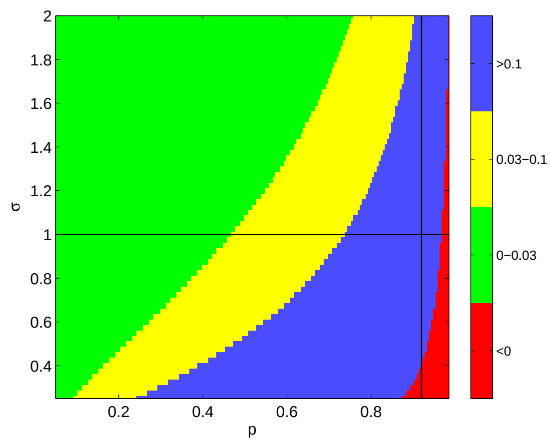
Figure 4: Tax multiplier on employment For Alternative Values of Risk Aversion and Price Adjustment Costs: No Technology Shock



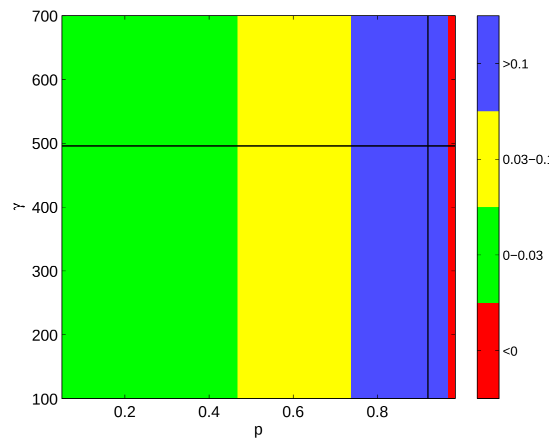
(a) Alternative values of p and σ NL solution.



(b) Alternative values of p and γ NL solution.



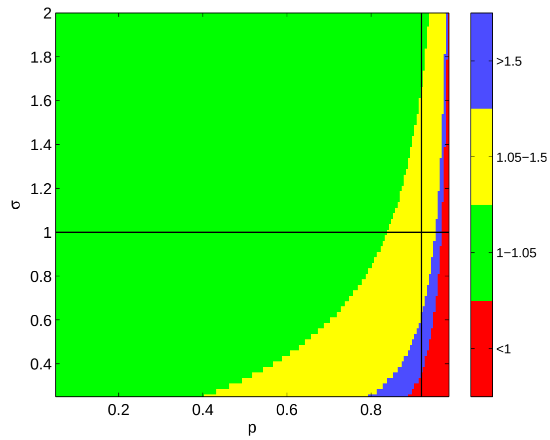
(c) Alternative values of p and σ LL solution.



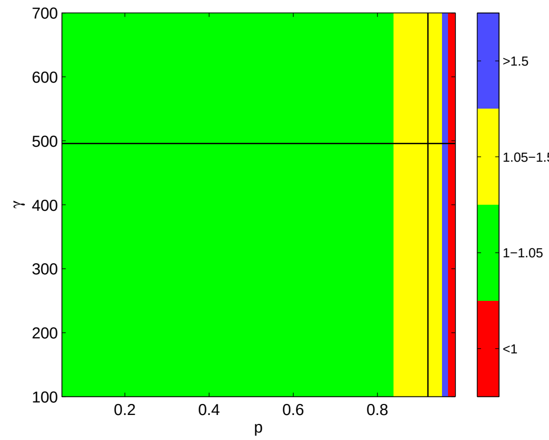
(d) Alternative values of p and γ LL solution.

Notes: The line denotes the baseline value of each parameter. Red: labor tax multiplier is negative (employment increases when the labor tax is cut); Green: labor tax multiplier is in $[0, 0.03]$; Yellow: labor tax multiplier is in $(0.03, 0.1]$; Blue: labor tax multiplier exceeds 0.1.

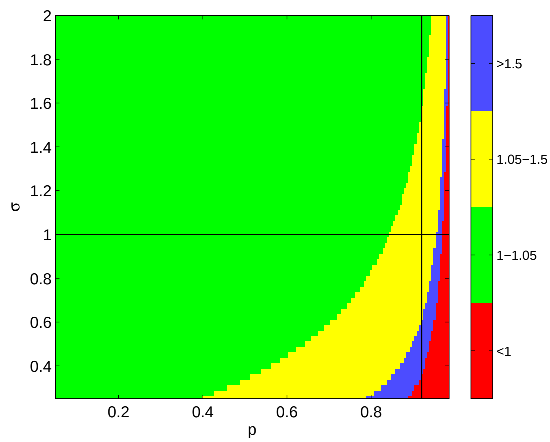
Figure 5: Government purchase multiplier on GDP For Alternative Values of Risk Aversion and Price Adjustment Costs: No Technology Shock



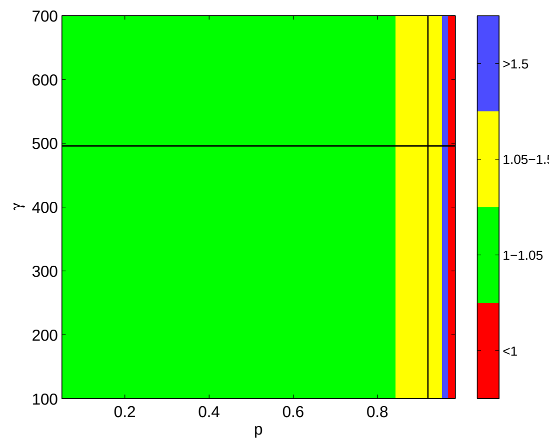
(a) Alternative values of p and σ NL Solution.



(b) Alternative values of p and γ NL solution.



(c) Alternative values of p and σ LL solution.



(d) Alternative values of p and γ LL solution.

Notes: The baseline parameterization is denoted with a line. Red: government-purchase-GDP-multiplier < 1; Green: the multiplier is in $[1, 1.05]$; Yellow: the multiplier is in $[1.05, 1.5]$; Blue: the multiplier exceeds 1.5.

A.3 Accounting for the Great Recession with the parameterization of Christiano and Eichenbaum (2012)

Christiano and Eichenbaum (2012) find that the government purchase multiplier exceeds 2 in a nonlinear Rotemberg model that is very close to ours.[3] In our model this can also occur but only in the neighborhood of the point where $slope(AD) = slope(AS)$. Moreover, in this neighborhood the sign and magnitudes of the fiscal multipliers are very sensitive to small perturbations of p and other structural parameters. It is thus interesting to investigate why their government purchase multipliers are so large.

Following their paper, we set the preference discount factor $\beta = 0.99$, the coefficient of relative risk aversion for consumption to $\sigma = 1$ and the curvature parameter for leisure to $\nu = 1$. The technology parameter θ that governs the substitutability of different types of goods is set to 3, the adjustment costs of price adjustment to $\gamma = 100$, and the conditional probability of exiting the low state to $p = 0.775$. The labor tax τ_w is set to 0.2, the government purchases share in output η to 0.2, and the subsidy to intermediate goods producers τ_s is set so that steady-state profits are zero. Finally, the coefficients on the Taylor rule are $\phi = 1.5$ and $\phi_y = 0$. With this parameterization our LL representation is identical to the one in Christiano and Eichenbaum (2012).[4]

Even though the LL equations are the same, the NL models are slightly different.[5] We first examine whether these differences matter by solving our nonlinear model using their parameter values. In turns out the differences in the two NL models are not important for the size of the fiscal multipliers. When we set $d^L = 1.0118$ and solve our model, the resulting government purchase multiplier for GDP is 2.2 using the NL solution which is the same value they report.

The reason the government purchase multiplier is so large using the Christiano and Eichenbaum (2012) parameterization is because the implied value for $slope(NKPC)$ is very large. Their parameterization implies that $slope(NKPC)$ is about 0.06 which is about three times larger than our baseline value of 0.021 and also well above 0.03 which is the top of the range of well identified estimates surveyed by Schorfheide (2008).[6] From equation (13) in

[3] Christiano, Eichenbaum, and Rebelo (2011) report similar results but it is easier for us to compare our results with the results of Christiano and Eichenbaum (2012) because they also posit Rotemberg adjustment costs.

[4] One difference between the models is that Christiano and Eichenbaum (2012) fix the level of government purchases as opposed to its share in output. However, the LL representations are equivalent when one considers the same type of fiscal policy shock.

[5] We assume that the resource costs of price adjustment apply to gross production ($\gamma\pi_t^2 y_t$) whereas they assume that the resource costs of price adjustment only apply to GDP ($\gamma\pi_t^2(c_t + g_t)$).

[6] Their parameterization implies $slope(NKPC) = 0.06$ when the share of government purchases in output is held fixed. If instead the level of government purchases is held fixed as they assume, $slope(NKPC)$ is

the paper we know that a larger value of $slope(NKPC)$ translates directly into a steeper AS schedule. A steeper AS schedule also results in a much larger inflation response to a d^L shock of a given size. In particular, their parameterization associates the 7% decline in output in the LL solution with a 7% decline in the annualized inflation rate. If we solve the model using the NL equilibrium conditions instead, output and the annualized inflation rate both fall by 5%. In fact, their value of $slope(NKPC)$ is so large that the model overstates the GR inflation target for all values of p using either the LL or the NL equilibrium conditions.[7]

Given how different their results are from ours we would like to adjust their parameterization so that the model can hit the two GR targets using the NL solution. One way to do this is to hold fixed their choices of the structural parameters and thus $slope(NKPC)$ and to use the technology shock to hit the inflation rate. Results reported in Figures 6–9 implement this scheme. From Figure 6 we see that under this calibration strategy their model parameterization ($\sigma = 1, p = 0.775$, and $\gamma = 100$) falls in the indeterminacy region. Their value of p is located just to the left of the point where $slope(AS) = slope(AD)$. When $\sigma = 1, p = 0.775$, and $\gamma = 100$ the targeted equilibrium exhibits $slope(AD) > slope(AS) > 0$ and the resulting government purchase multiplier is 4.6. The non-targeted ZLB equilibrium exhibits $slope(AS) > slope(AD) > 0$ and there is a third equilibrium with a positive interest rate as shown in Figure 7. In this neighborhood the local properties of equilibrium and thus the signs and magnitudes of the fiscal multipliers are very sensitive to the precise choice of parameters in this region of the parameter space. For instance, if p is increased from 0.775 to 0.79, the equilibrium switches to a Sunspot ZLB equilibrium and the government purchase multiplier is -10.0.

Christiano and Eichenbaum (2012) do not allow for technology shocks. To see whether this difference matters, we recalibrated the model to reproduce the GR facts with $z^L = 1$. It is not possible to reproduce the GR facts using their parameterization and adjusting d^L and θ because the resulting value of θ is less than 1 and thus not economically meaningful. However, it is possible with $\gamma = 300$. The resulting value of θ is small, 1.24, but it exceeds 1.

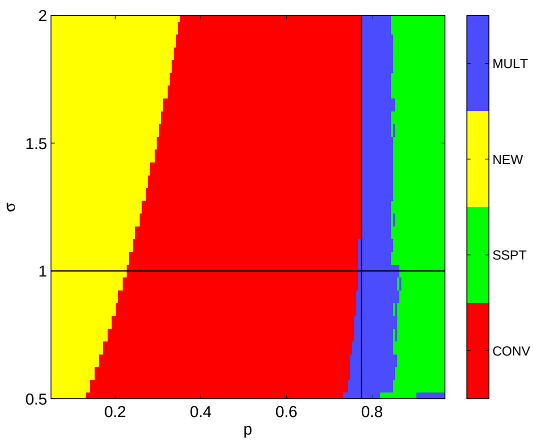
Figures 10-12 report results using the no-technology-shock calibration scheme. The no-technology shock calibration scheme results in lower fiscal multipliers at larger values of p because $slope(AS)$ is independent of p for the reasons described above.[8] For instance, the government purchase multiplier is now 1.06 at their baseline value of $p = 0.775$, as compared to 4.6 in the specification with technology shocks. In fact, the government purchase multiplier only exceeds 1.5 when $p \in [0.94, 0.965]$ but for these settings of p, $\theta < 1$ (see Figure 10).

0.0675.

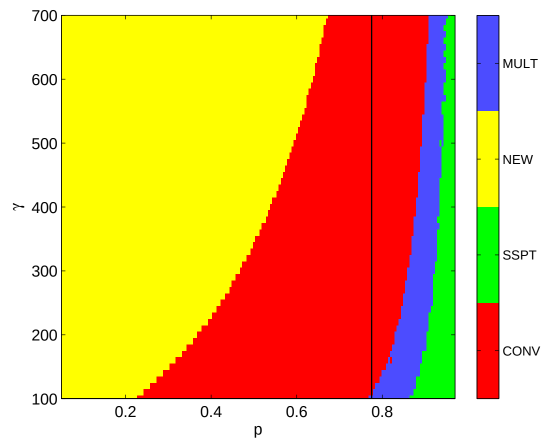
[7] This result also occurs if we set d^L to produce a 7% decline in output using the NL equilibrium conditions.

[8] The value of $slope(AS)$ is 0.0342.

Figure 6: Types of ZLB Equilibria for Alternative Values of Risk Aversion and Price Adjustment Costs: Christiano-Eichenbaum (2012) Parameterization with Technology Shock



(a) Alternative Combinations of p and σ.



(b) Alternative Combinations of p and γ.

Notes: The line denotes the baseline value of each parameter. The baseline value of γ is 100. Red: Conventional ZLB equilibrium (slope(AD)>slope(AS)>0); Green: Sunspot ZLB equilibrium (slope(AS)>slope(AD)>0); Yellow: New ZLB equilibrium (slope(AD)<0<slope(AS)); Blue: Multiple ZLB equilibria.

Figure 7: Multiple ZLB equilibrium at $p = 0.775$: Christiano-Eichenbaum (2012) Parameterization with Technology Shock

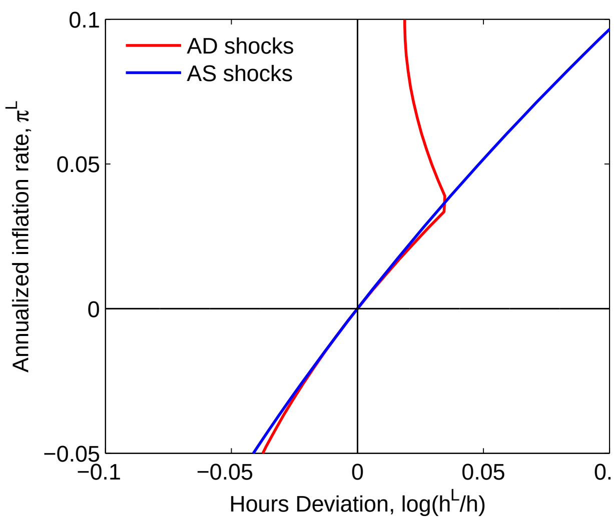
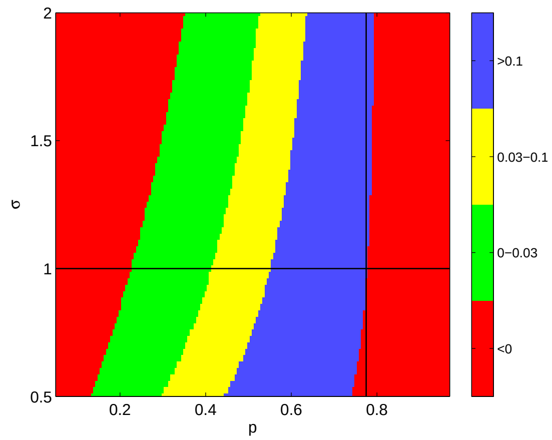
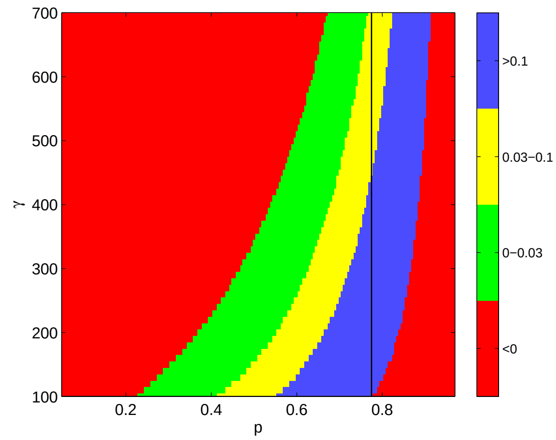


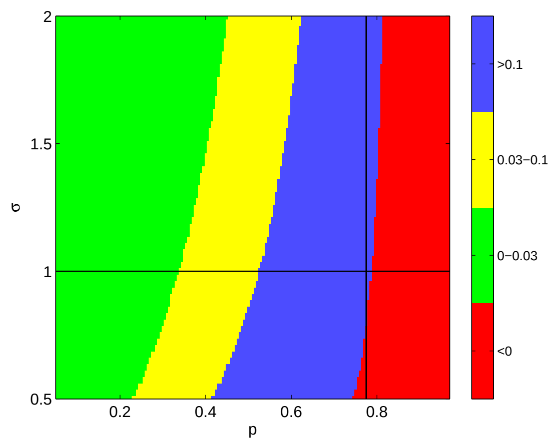
Figure 8: Labor Tax Multiplier on Employment for Alternative Values of Risk Aversion and Price Adjustment Costs: Christiano-Eichenbaum (2012) Parameterization with Technology Shock



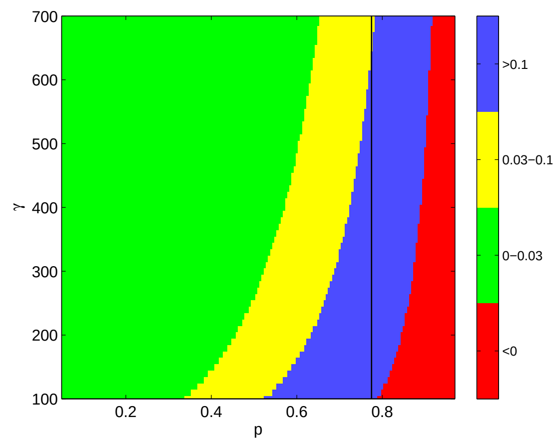
(a) Alternative values of p and σ NL solution.



(b) Alternative values of p and γ NL solution.



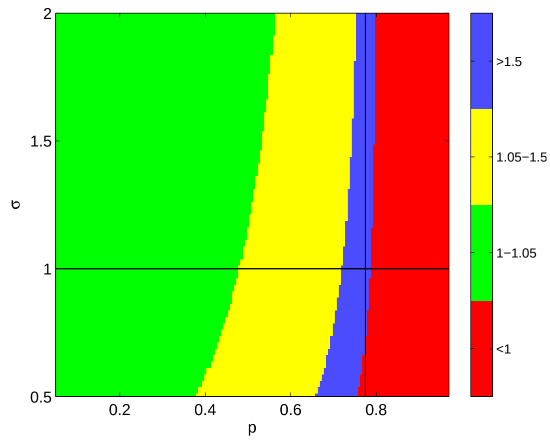
(c) Alternative values of p and σ LL solution.



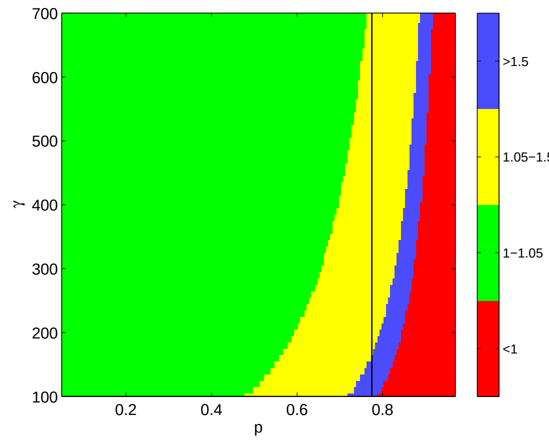
(d) Alternative values of p and γ LL solution.

Notes: The line denotes the baseline value of each parameter. The baseline value of γ is 100. Red: labor tax multiplier is negative (employment increases when the labor tax is cut); Green: labor tax multiplier is in $[0, 0.03]$; Yellow: labor tax multiplier is in $(0.03, 0.1]$; Blue: labor tax multiplier exceeds 0.1.

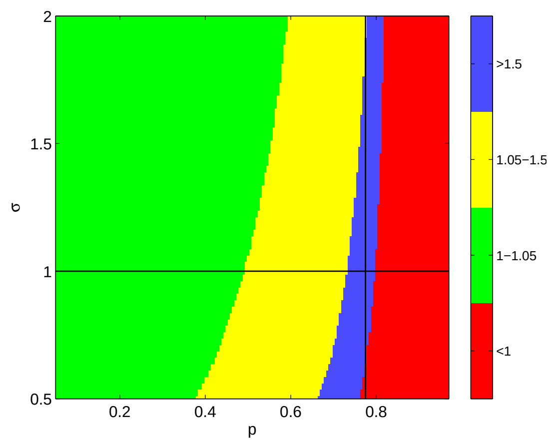
Figure 9: Government purchase multiplier on GDP For Alternative Values of Risk Aversion and Price Adjustment Costs: Christiano-Eichenbaum (2012) Parameterization with Technology Shock



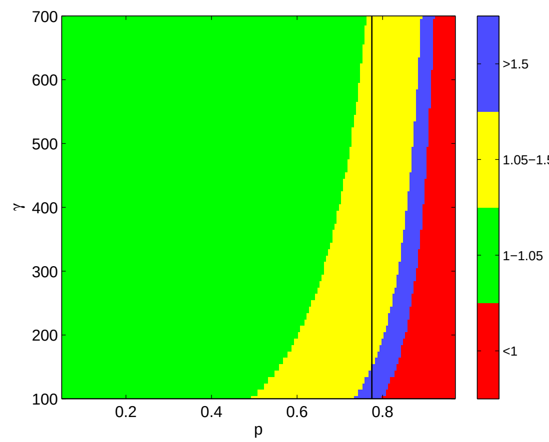
(a) Alternative values of p and σ NL Solution.



(b) Alternative values of p and γ NL solution.



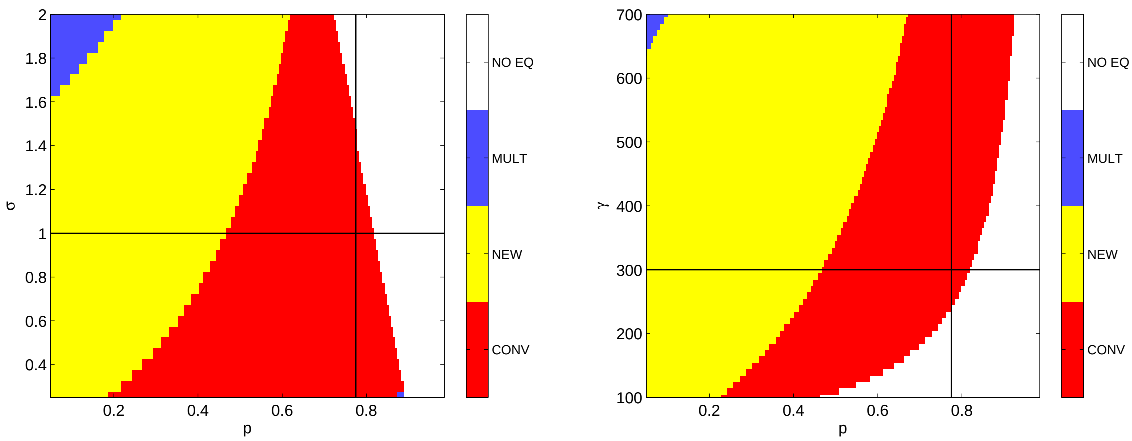
(c) Alternative values of p and σ LL solution.



(d) Alternative values of p and γ LL solution.

Notes: The baseline parameterization is denoted with a line. The baseline value of γ is 100. Red: government-purchase-GDP-multiplier < 1; Green: the multiplier is in $[1, 1.05]$; Yellow: the multiplier is in $[1.05, 1.5]$; Blue: the multiplier exceeds 1.5.

Figure 10: Types of ZLB Equilibria For Alternative Values of Price Adjustment Costs and Risk Aversion: Christiano-Eichenbaum (2012) Parameterization without technology shock



(a) Alternative Combinations of p and σ.

(b) Alternative Combinations of p and γ.

Notes: The baseline parameterization is denoted with a line. Red: Conventional ZLB equilibrium (slope(AD)>slope(AS)>0); Green: Sunspot ZLB equilibrium (slope(AS)>slope(AD)>0); Yellow: New ZLB equilibrium (slope(AD)<0<slope(AS)); Blue: Multiple ZLB equilibria; White: $\theta < 1$.

The no-technology-shock results have several other noteworthy features. Now Multiple ZLB equilibria occur at higher values of σ even when p is small and far away from the asymptote. The targeted ZLB equilibrium in this region continues to have $slope(AD) < 0$ and $slope(AS) > 0$ and it follows that the labor tax multiplier has an orthodox sign in the entire region with Multiple ZLB equilibria (Figure 11).[9] However, the overall size of the region with a downward-sloping AD schedule is smaller in Figure 10 as compared to our baseline calibration without technology shocks reported in the left panel of Figure 3. This is because the value of $\gamma = 300$ is lower than our baseline value of 495.8.

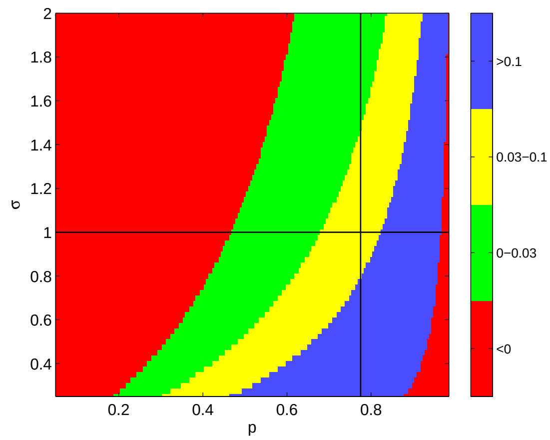
A.4 Accounting for the Great Recession with the parameterization of Denes, Eggertsson and Gilbukh (2013)

We now consider the parameterization of Denes, Eggertsson, and Gilbukh (2013). Their estimated parameterization is interesting because their estimates imply a much smaller value of $slope(NKPC) = 0.0075$ than we have considered up to this point.

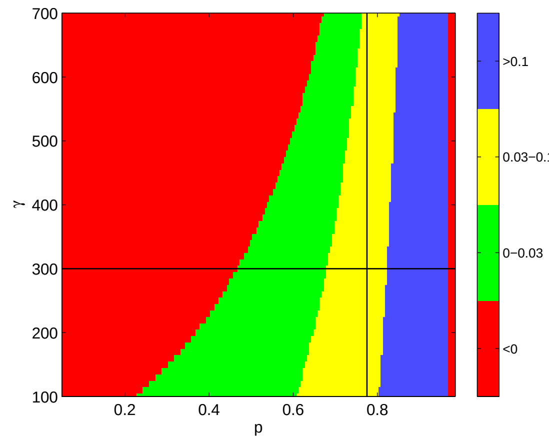
Denes, Eggertsson, and Gilbukh (2013) consider a NK framework with Calvo price adjustment and firm specific labor markets and a single shock to d^L. This is a different model

[9] In the non-target equilibrium inflation and output exceed their steady-state levels but it is still a ZLB equilibrium because d^L is large.

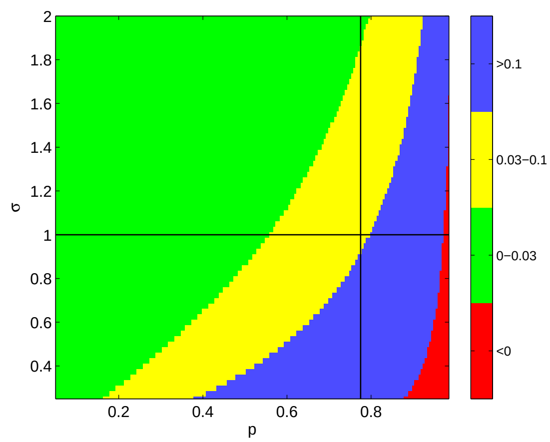
Figure 11: Tax multiplier on Employment for Alternative Values of Risk Aversion: Christiano-Eichenbaum (2012) Parameterization without technology shock



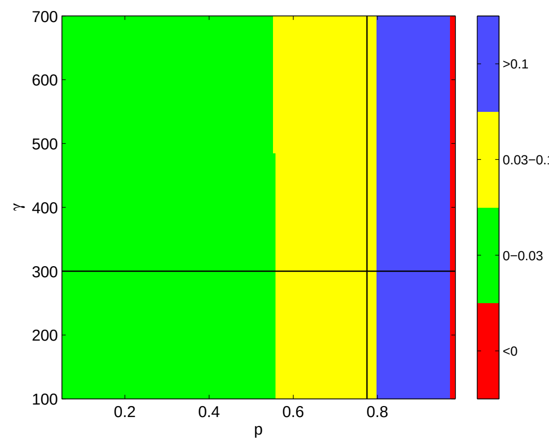
(a) Alternative values of p and σ NL solution.



(b) Alternative values of p and γ NL solution.



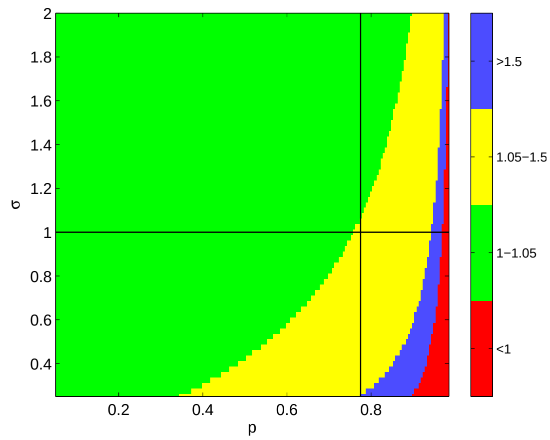
(c) Alternative values of p and σ LL solution.



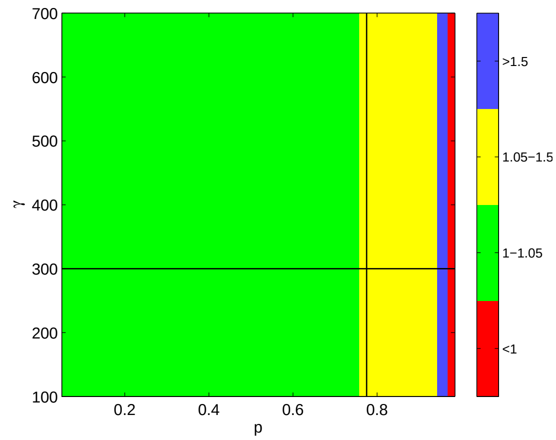
(d) Alternative values of p and γ LL solution.

Notes: The line denotes the baseline value of each parameter. Red: labor tax multiplier is negative (employment increases when the labor tax is cut); Green: labor tax multiplier is in $[0, 0.03]$; Yellow: labor tax multiplier is in $(0.03, 0.1]$; Blue: labor tax multiplier exceeds 0.1.

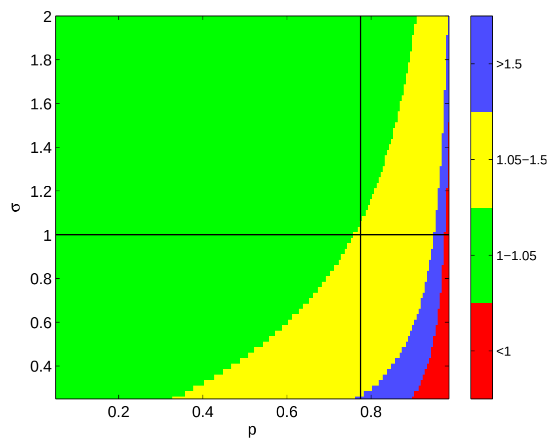
Figure 12: Government purchase multiplier on GDP For Alternative Values of Risk Aversion: Christiano-Eichenbaum (2012) Parameterization without technology shock



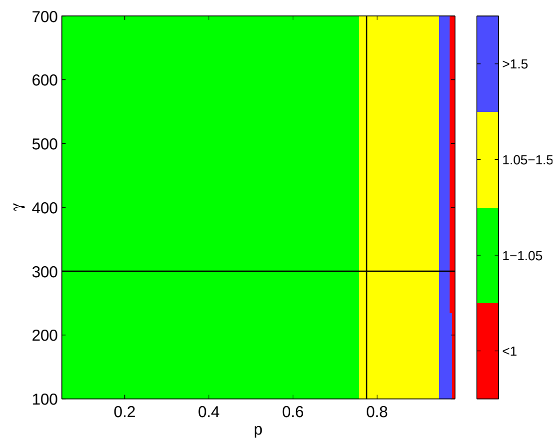
(a) Alternative values of p and σ NL Solution.



(b) Alternative values of p and γ NL solution.



(c) Alternative values of p and σ LL solution.



(d) Alternative values of p and γ LL solution.

Notes: The line denotes the baseline value of each parameter. Red: government-purchase-GDP-multiplier < 1; green: the multiplier is in $[1, 1.05]$; Yellow: the multiplier is in $[1.05, 1.5]$, Blue: the multiplier exceeds 1.5.

from ours but they solve their model using a LL solution method and the LL equilibrium conditions are identical to ours. They estimate their model parameters using an overidentified Quasi-Bayesian method of moments procedure with two moments that they associate with the GR: an output decline of 10% and an (annualized) decline in the inflation rate of -2%. Their estimates are: $(p, \theta, \beta, \sigma, \nu) = (0.857, 13.23, 0.997, 1.22, 1.69)$. Finally, their fiscal parameters are set in the same way that we have assumed up to now.

We are interested in understanding the properties of our model in this region of the parameter space. However, in order to do that we must first make some small adjustments to their estimates. Our practice has been to calibrate the model using our *nonlinear* equilibrium conditions. Here we adjust d^L and γ to reproduce our GR inflation and output targets using our NL equilibrium conditions.[10] The resulting value of γ is 6341. These adjustments have only a very small effect on $slope(NKPC)$. It rises from 0.0075 using the estimated parameterization of Denes, Eggertsson, and Gilbukh (2013) to 0.0086 using our calibrated values of d^L and γ.

Why is γ so large for this parameterization? First, their estimate of $p = 0.857$ is large. In Section E we show that reproducing the GR declines in output and inflation requires a small value of $slope(NKPC)$ when p is large. Indeed, the value of $slope(NKPC)$ here is less than half the size implied by our baseline calibration and this requires a large value of γ. Second, their estimates of θ, σ, and ν are also much higher than our estimates. For given γ higher values of these other parameters increase $slope(NKPC)$. Thus, producing a small $slope(NKPC)$ requires a particularly large value of γ (See Section E for more details).

The most noteworthy new property of the model is shown in Figure 13.[11] The region where the New ZLB equilibrium occurs is now much smaller and instead there is a very large region with Multiple ZLB equilibria at low values of p. The size of this region increases with σ and γ. Throughout this region there are two ZLB equilibria, the targeted equilibrium has $slope(AD) < 0 < slope(AS)$ and the second has $0 < slope(AD) < slope(AS)$.

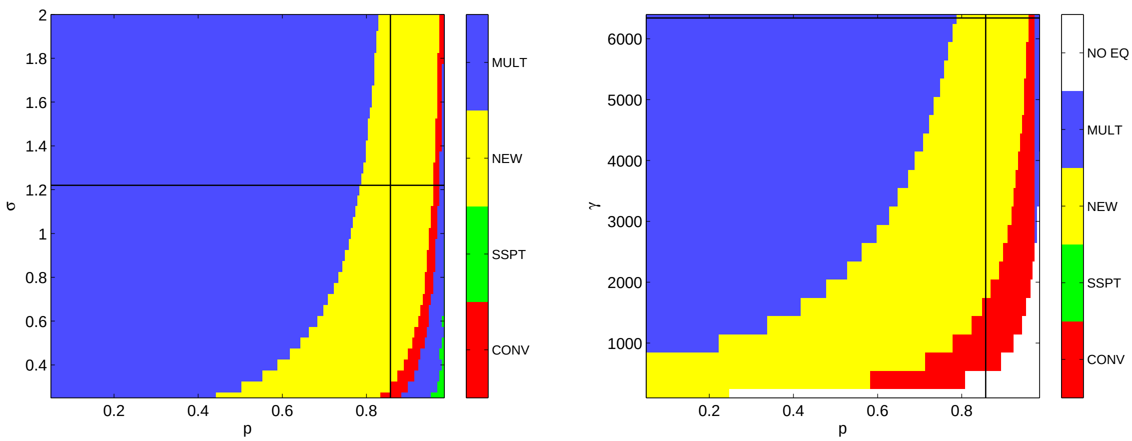
The model produces a New ZLB equilibrium using the value of $p = 0.857$, estimated by Denes, Eggertsson, and Gilbukh (2013). Employment increases by 0.20 percent in response to a one percentage point reduction in the labor tax (Figure 14). Using the LL solution one infers instead that the equilibrium is Conventional and employment falls by 0.11 percent instead.

The size of the government purchase multiplier using the NL equilibrium conditions is 1.08

[10] The value of γ implied by their estimated reduced form is very large and calibrating our model in this way brings the value of γ down a bit.

[11] To conserve on space we only report figures using the no-technology-shock calibration scheme for this parameterization of our model.

Figure 13: Types of ZLB Equilibria for Alternative Values of Risk Aversion and Price Adjustment Costs: Denes et al. (2013). Parameterization with No Technology Shocks.



(a) Alternative Combinations of p and σ.

(b) Alternative Combinations of p and γ.

Notes: The line denotes the baseline value of each parameter. The baseline value of γ is 6341. Red: Conventional ZLB equilibrium (slope(AD)>slope(AS)>0); Green: Sunspot ZLB equilibrium (slope(AS)>slope(AD)>0); Yellow: New ZLB equilibrium (slope(AD)<0<slope(AS)); Blue: Multiple ZLB equilibria; White: $\theta < 1$.

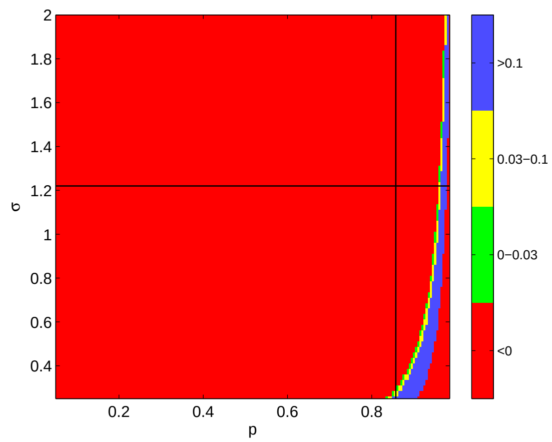
at the reference parameterization and with marginally higher σ it falls below 1.05 (Figure 15). There are some larger differences between the NL and LL government purchase multipliers here. The LL solution produces larger government purchase multipliers at lower values of p. However, once again we see that a government purchase multiplier in excess of 1.5 only occurs in a very small region of the parameter space as indexed by p, σ and γ.

To summarize, the results we have reported here are consistent with the message of our paper. The NK model may also exhibit orthodox and small fiscal multipliers at the ZLB using parameterizations of the model considered by the previous literature and shocks that are set to reproduce the GR declines in GDP and inflation.

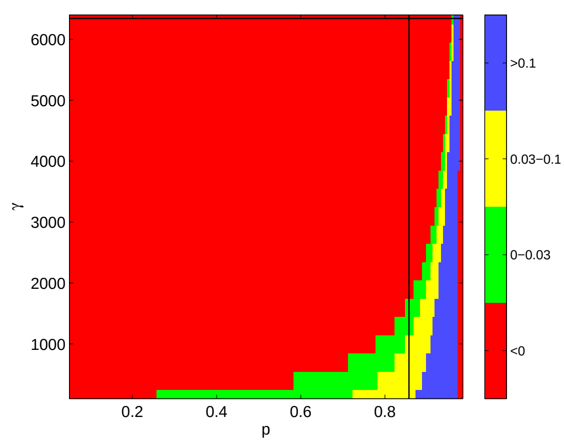
A.5 Great Depression

This subsection illustrates the biases in the LL solution can be much larger if the NK model is calibrated to the Great Depression (GD) instead. We use a parameterization of the model based on Eggertsson (2011) who analyzes fiscal multipliers in the GD using the LL solution of a NK model with Calvo price setting and firm specific labor. That model has the same LL representation as our model. It is thus interesting to investigate the biases in the LL solution using his parameterization. Row 1 shows the fiscal multipliers for the LL and

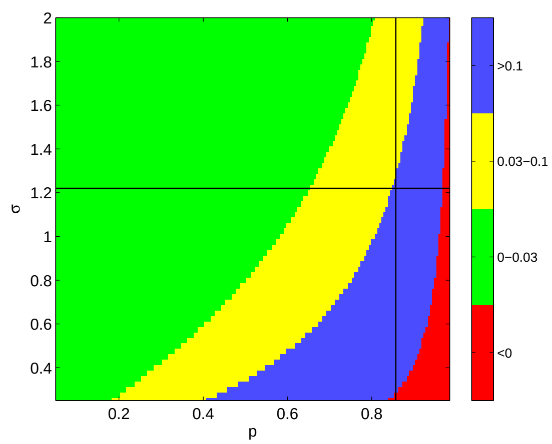
Figure 14: Tax Multiplier on Employment for Alternative Values of Risk Aversion and Price Adjustment Costs: Denes et al. (2013). Parameterization with No Technology Shocks



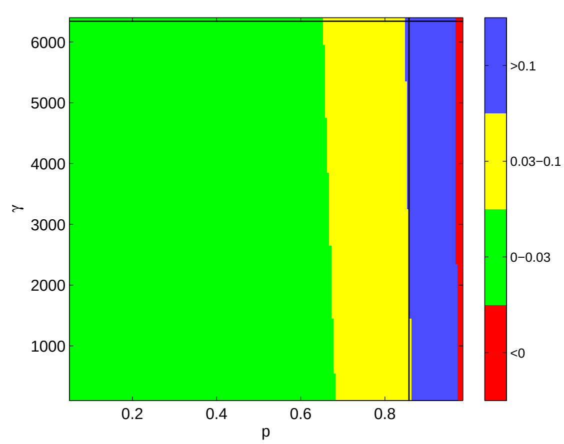
(a) Alternative values of p and σ NL solution.



(b) Alternative values of p and γ NL solution.



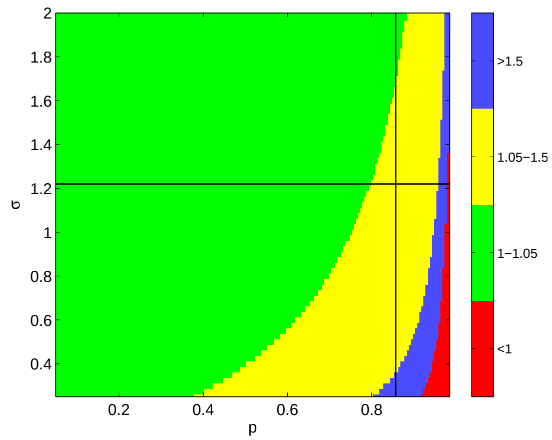
(c) Alternative values of p and σ LL solution.



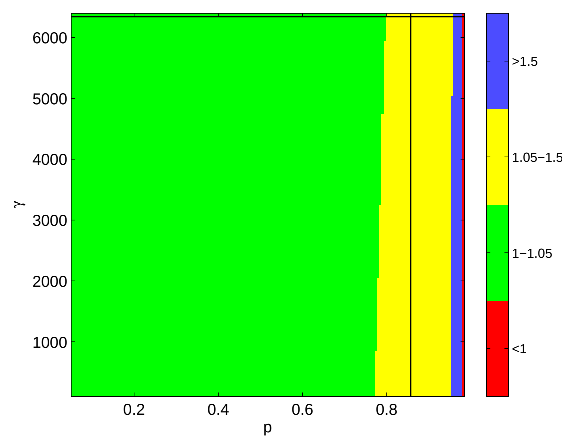
(d) Alternative values of p and γ LL solution.

Notes: The line denotes the baseline value of each parameter. The baseline value of γ is 6341. Red: labor tax multiplier is negative (employment increases when the labor tax is cut); Green: labor tax multiplier is in $[0, 0.03]$; Yellow: labor tax multiplier is in $(0.03, 0.1]$; Blue: labor tax multiplier exceeds 0.1.

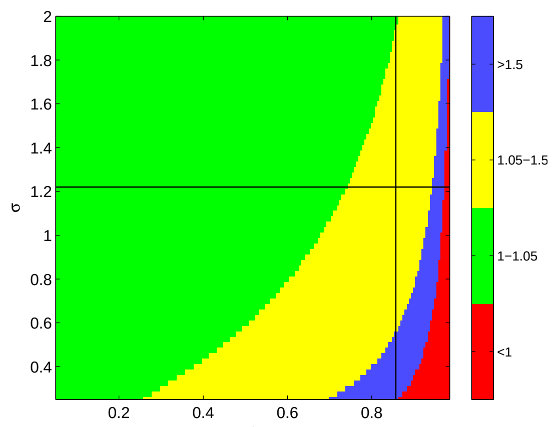
Figure 15: Government purchase multiplier on GDP for Alternative Values of Risk Aversion and Price Adjustment Costs: Denes et al. (2013) Parameterization with No Technology Shocks, Nonlinear (Top) vs. Loglinear (Bottom)



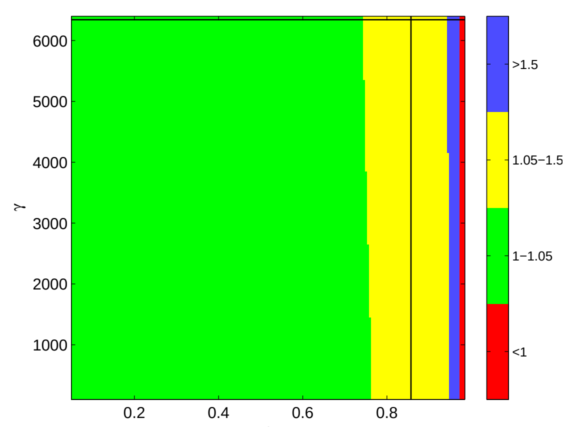
(a) Alternative values of p and σ NL Solution.



(b) Alternative values of p and γ NL solution.



(c) Alternative values of p and σ LL solution.



(d) Alternative values of p and γ LL solution.

Notes: The line denotes the baseline value of each parameter. The baseline value of γ is 6341. Red: government-purchase-GDP-multiplier < 1; Green: the multiplier is in $[1, 1.05]$; Yellow: the multiplier is in $[1.05, 1.5]$; Blue: the multiplier exceeds 1.5.

NL solutions of our model using shocks and parameters taken from Eggertsson (2011).[12] The government purchase GDP multiplier is 1.82 using the LL solution but is only 1.2 using the NL solution. The type of equilibrium differs across the two solutions with the LL solution producing a Conventional configuration and the NL solution an equilibrium with $slope(AS) < slope(AD) < 0$. This type of ZLB equilibrium did not occur for the GR parameterizations but becomes a relevant possibility for shocks of this magnitude.

The model has been calibrated to produce 30% decline in GDP and a 10% decline in the annualized inflation rate using the LL solution. The declines in GDP and inflation are much smaller using the NL solution. GDP falls by 24% and the annualized inflation rate falls by only 0.03%.

If one uses the exact solution is to reproduce the GD instead, the resulting configuration of shocks and parameters is very different. To illustrate this point, we calibrated the model

Table 1: ZLB Equilibria Great Depression

Specification	d^L	κ^L	Slope(AD)	Slope(AS)	$\frac{\Delta GDP}{\Delta G}$	$\frac{\Delta h}{\Delta \tau_w}$
1) Calibrated using LL solution,	1.0134					
NL solution		0.11	-0.033	-0.22	1.20	-0.54
LL solution		1.24	0.12	0.086	1.82	1.02
2) Calibrated using NL solution,	1.0213					
NL solution		0.41	-0.015	-0.032	1.26	-0.87
LL solution		N.A.	N.A.	N.A.	N.A.	N.A.

The values of κ^L for the LL solution are imputed.

using the NL solution to hit the GD targets by adjusting d^L and γ instead.[13] Results for this calibration of the model are reported in row 2 of Table 1.

Using the NL equilibrium conditions to calibrate the model results in a ZLB equilibrium with $slope(AS) < slope(AD) < 0$ and there is also a positive interest rate equilibrium. The government purchase GDP multiplier is 1.26 in the ZLB equilibrium and the labor tax multiplier is negative: a one percentage point increase in the labor tax lowers labor input by 0.87%.

The breakdown in the LL solution is particularly severe in this example. According to the LL equilibrium conditions, there is no ZLB equilibrium for this parameterization of the

[12]The parameterization of the model is: $\beta = 0.9970$, $\sigma = 1.16$, $p = 0.903$, $\theta = 12.77$, $\nu = 1.5692$ and $\gamma = 4059.8$. This choice of γ pins down the correct value of $slope(NKPC)$. There is only a single shock to d^L and its value is 1.0134.

[13]The resulting values are $d^L = 1.0213$ and $\gamma = 1209.2$. We considered several other strategies for calibrating the exact model to reproduce a 30% decline in GDP and a 10% decline in inflation using the same values of structural parameters including γ. We found that no combination of d^L and p worked. In fact, even if we allowed for d^L, p and $z^L < z$, the Rotemberg model still could not hit these two targets.

model. Thus, we can see that for large shocks such as the GD, LL solutions exhibit large biases in the government purchase multiplier and may provide incorrect inference about the existence of a ZLB equilibrium.

Appendix B Calvo model with a single labor market

This section presents the equilibrium conditions in the Calvo model with a single labor market. They are given by

$$
\begin{aligned}
c_t^{\sigma} h_t^{\nu} &= w_t(1-\tau_{w,t}), \\
1 &= \beta d_{t+1} E_t \left\{ \frac{1+R_t}{1+\pi_{t+1}} \left(\frac{c_t}{c_{t+1}} \right)^{\sigma} \right\} \\
gdp_t &= \frac{1}{x_t} z_t h_t, \\
c_t &= (\frac{1}{x_t} - \eta_t) z_t h_t, \\
as_{1,t} &= gdp_t + \beta\alpha d_{t+1} E_t[\frac{c_{t+1}^{-\sigma}}{c_t^{-\sigma}} (1+\pi_{t+1})^{\theta-1} as_{1,t+1}], \\
as_{2,t} &= \frac{c_t^{\sigma} h_t^{\nu}}{(1-\tau_{w,t})} \frac{gdp_t}{z_t} + \beta\alpha d_{t+1} E_t[\frac{c_{t+1}^{-\sigma}}{c_t^{-\sigma}} (1+\pi_{t+1})^{\theta} as_{2,t+1}], \\
\tilde{P}_t &= \frac{as_{2,t}}{as_{1,t}}, \\
x_t &= (1-\alpha)\tilde{P}_t^{-\theta} + \alpha(1+\pi_t)^{\theta} x_{t-1}, \\
1 &= (1-\alpha)\tilde{P}_t^{1-\theta} + \alpha(1+\pi_t)^{\theta-1} \\
R_t &= \max(0, r_t^e + \phi_\pi \pi_t + \phi_y \widehat{gdp}_t)
\end{aligned}
$$

where $\tilde{P}_t$ is the real price which is chosen by firms that can change their nominal prices at time t, x_t summarizes the relative price dispersion, and α is the probability that a firm is unable to change its price. The term $1/x_t$ introduces a wedge between GDP and gross output ($z_t h_t$), and $1 - 1/x_t$ acts as κ_t in our baseline Rotemberg model. In NK models with Calvo price setting, the distribution of relative prices is non-degenerate. It follows that the allocation of factors of production (labor in this model) is inefficient. In other words, higher price dispersion reduces GDP compared to the production level that is possible if all relative prices are 1.

Because x_t is a state variable, the equilibrium conditions cannot be summarized by the AD and the AS schedules without further assumptions. We make the following assumption: x is constant at x^L when the shocks are (d^L, z^L) and becomes 1 immediately after the shocks dissipate. This allows us to use the AD and the AS schedules to characterize the ZLB equilibrium using Calvo price setting.

Once the shocks dissipate, all variables jump to the zero inflation steady-state, where $x = 1$, $gdp = h = \{(1-\tau_w)/(1-\eta)^\sigma\}^{1/(\sigma+\nu)}$, $as_1 = h/(1-\beta\alpha)$, $as_2 = (1-\eta)^\sigma h^{1+\sigma+\nu}/\{(1-\beta\alpha)(1-\tau_w)\}$, and $\tilde{P} = 1$. In the L state, by assumption, x^L can be written as

$$x^L = \frac{(1-\alpha)(\tilde{P}^L)^{-\theta}}{1-\alpha(1+\pi^L)^\theta} = \frac{1-\alpha}{1-\alpha(1+\pi^L)^\theta}\left\{\frac{1-\alpha(1+\pi^L)^{\theta-1}}{1-\alpha}\right\}^{\frac{\theta}{\theta-1}}.$$

The AD schedule is identical to the AD schedule in the Rotemberg model, except that the term κ^L is now equal to $(x^L-1)/x^L$ (compare with equation (12) in the main paper).

The AS schedule is a little bit more complicated. First observe

$$\tilde{P}^L = \frac{(c^L)^{-\sigma}as_2^L}{(c^L)^{-\sigma}as_1^L}.$$

Then using

$$(c^L)^{-\sigma}as_1^L = (c^L)^{-\sigma}gdp^L + \beta\alpha d^L\left\{p(c^L)^{-\sigma}(1+\pi^L)^{\theta-1}as_1^L + (1-p)c^{-\sigma}as_1\right\}$$

and

$$(c^L)^{-\sigma}as_2^L = \frac{(c^L)^\sigma(h^L)^\nu}{(1-\tau_w^L)}\frac{gdp^L}{z^L} + \beta\alpha d^L\left\{p(c^L)^{-\sigma}(1+\pi^L)^\theta as_2^L + (1-p)c^{-\sigma}as_2\right\},$$

we obtain $\tilde{P}^L = G(h^L, \pi^L)$.[14] Note next that

$$\tilde{P}^L = \left\{\frac{1-\alpha(1+\pi^L)^{\theta-1}}{1-\alpha}\right\}^{\frac{1}{1-\theta}},$$

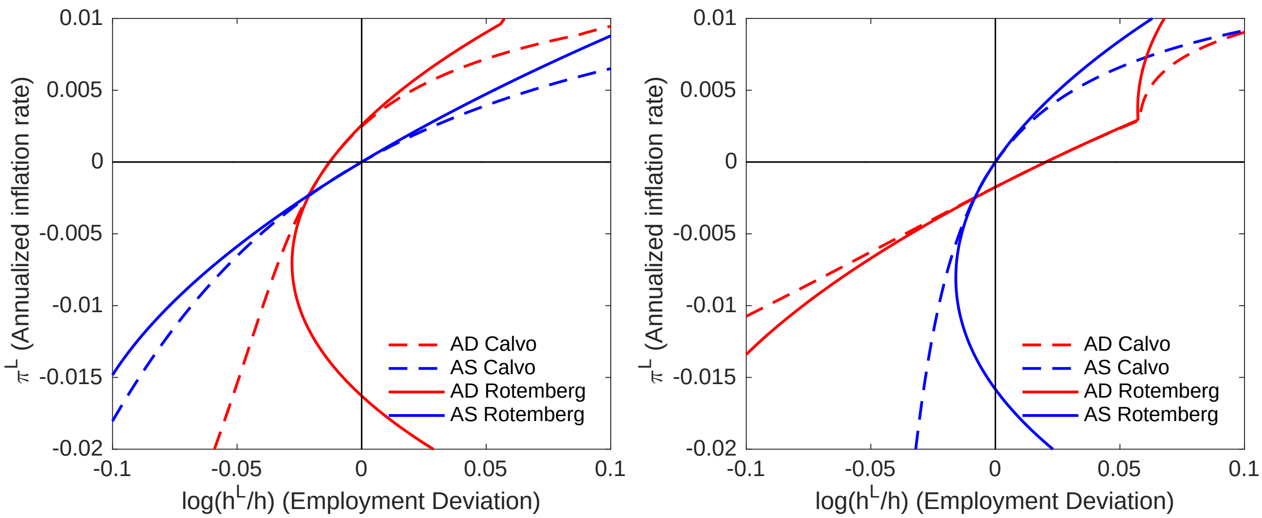
and the AS schedule is thus given by

$$\left\{\frac{1-\alpha(1+\pi^L)^{\theta-1}}{1-\alpha}\right\}^{\frac{1}{1-\theta}} = G(h^L, \pi^L).$$

Figure 16 shows the AD-AS schedules for $p = 0.8$ and $p = 0.92$. For both choices of p the AD and the AS schedules in both models are upward sloping and the local dynamics are the same. In both models the equilibrium is Conventional in the former case and a Sunspot equilibrium in the later case.

[14] This is because both c^L and gdp^L can be expressed by x^L, h^L, and exogenous variables and parameters, and because x^L is a function of π^L.

Figure 16: ZLB Equilibria in the Calvo vs. Rotemberg model



(A) $p = 0.8$ (B) $p = 0.92$

Appendix C Existence of a ZLB equilibrium using the LL equilibrium conditions

To make the arguments more transparent we will assume that $\hat{\eta}^L = \hat{\tau}_w^L = 0$.

Proposition 1 Existence of a ZLB equilibrium using the LL equilibrium conditions. *Suppose* $\hat{\eta}^L = \hat{\tau}_w^L = 0$, $(\phi_\pi, \phi_y) \geq (p, 0)$, $\hat{d}^L \geq 0$, $\hat{z}^L \leq 0$, $0 < p \leq 1$, $\sigma \geq 1$, *and that* AD^{LL} *and* AS^{LL} *do not coincide in state L. Then there exists a unique ZLB equilibrium with deflation and depressed labor input,* $(\pi^L, \hat{h}^L) < (0, 0)$, *if*

Conventional ZLB equilibrium

1a) $slope(AD^{LL}) > slope(AS^{LL})$ *and*

1b) $\left(slope(AD^{LL}) - slope(AS^{LL})\frac{\sigma-1}{\sigma+\nu}\right)\hat{z}^L - \frac{\hat{r}_L^e}{p} > 0,$

or

Sunspot ZLB equilibrium

2a) $slope(AD^{LL}) < slope(AS^{LL})$ *and*

2b) $\left(slope(AD^{LL}) - slope(AS^{LL})\frac{\sigma-1}{\sigma+\nu}\right)\hat{z}^L - \frac{\hat{r}_L^e}{p} < 0.$

If the parameters do not satisfy either both 1a) and 1b) or alternatively both 2a) and 2b), then there is no ZLB equilibrium with depressed labor input $\hat{h}^L < 0$.[15]

Proof The AD and AS schedules are

$$\pi^L = [slope(AD^{LL})\hat{z}^L - \frac{\hat{r}_L^e}{p}] + slope(AD^{LL})\hat{h}^L, \text{and } \pi^L = slope(AS^{LL})\frac{\sigma-1}{\sigma+\nu}\hat{z}^L + slope(AS^{LL})\hat{h}^L.$$

They are upward-sloping, because $slope(AD^{LL})$ and $slope(AS^{LL})$ are positive.

First, assume (1a) and (1b). Then AD^{LL} is strictly steeper than AS^{LL}. Moreover, the intercept term of AD^{LL} is larger than the intercept of AS^{LL}. It follows that at their intersection $\hat{h}^L < 0$. Solving for π^L, we obtain

$$\pi^L = \frac{slope(AS^{LL})}{slope(AS^{LL}) - slope(AD^{LL})}\left[slope(AD^{LL})\hat{z}^L + slope(AD^{LL})\frac{\sigma-1}{\sigma+\nu}(-\hat{z}^L) - \frac{\hat{r}_L^e}{p}\right].$$

Since $slope(AS^{LL}) - slope(AD^{LL}) < 0$, π^L is negative because

$$\begin{aligned}
& slope(AD^{LL})\hat{z}^L + slope(AD^{LL})\frac{\sigma-1}{\sigma+\nu}(-\hat{z}^L) - \frac{\hat{r}_L^e}{p} \\
\geq \; & slope(AD^{LL})\hat{z}^L + slope(AS^{LL})\frac{\sigma-1}{\sigma+\nu}(-\hat{z}^L) - \frac{\hat{r}_L^e}{p} \\
& \quad \text{(By assumption, } (\sigma-1)(-\hat{z}^L) \geq 0 \text{ and } slope(AS^{LL}) - slope(AD^{LL}) < 0.) \\
> \; & 0. \quad \text{(By condition 1a).}
\end{aligned}$$

Thus, at the intersection of the AD and the AS schedules, $(\pi^L, \hat{h}^L) < (0, 0)$.

What remains to show is that at the intersection, the Taylor rule implies a zero nominal interest rate. The linear part of the Taylor rule prescribes

$$\begin{aligned}
& \hat{r}^e + \phi_\pi \pi^L + \phi_y \hat{gdp}^L \\
< \; & p\left(slope(AD^{LL}) - slope(AS^{LL})\frac{\sigma-1}{\sigma+\nu}\right)\hat{z}^L + \phi_\pi \pi^L + \phi_y \hat{gdp}^L. \quad \text{(Condition 1a).} \quad (1)
\end{aligned}$$

Since $\hat{gdp}^L = \hat{z}^L + \hat{h}^L$, we know that $(\hat{z}^L, \pi^L, \hat{g}dp^L)$ are all negative. Condition 1b) implies

[15] The final statement leaves open the possibility that a ZLB equilibrium with $\hat{h}^L \geq 0$ exists for parameterizations that satisfy 1a) and 2b) (or 1b) and 2a)). This is possible when z^L is sufficiently low. If it is assumed that $\hat{z}^L = 0$, then the final clause can be removed and *any* ZLB equilibrium must satisfy both (1a) and (1b) or both (2a) and (2b).

that the coefficient on $\hat{z}^L$ is strictly positive. Combining this result with the assumption $(\phi_\pi, \phi_y) \geq 0$, it follows that the RHS of (1) is strictly negative, and the nominal interest rate at the intersection is zero.

Next, assume (2a) and (2b) instead. The proof for this case is almost identical to the case with (1a) and (1b). The only difference is that ϕ_π needs to be sufficiently large to have $\hat{r}^e_L + \phi_\pi \pi^L + \phi_y \hat{y}^L < 0$. Since AD^{LL} is upward sloping and $\hat{h}^L < 0$, π^L is smaller than the intercept of AD^{LL}, which is given by $slope(AD^{LL})\hat{z}^L - \hat{r}^e_L/p$. Thus, the assumption $\phi_\pi \geq p$ implies

$$\hat{r}^e_L + \phi_\pi \pi^L + \phi_y \hat{gdp}^L \leq \hat{r}^e_L + p\pi^L \leq \hat{r}^e_L + p\{slope(AD^{LL})\hat{z}^L - \hat{r}^e_L/p\} \leq p \times slope(AD^{LL})\hat{z}^L \leq 0.$$

It follows that the nominal interest rate is zero.

Finally, suppose 1b) holds but 1a) doesn't. Then AD^{LL} is not steeper than AS^{LL}, and the intercept of the AD^{LL} is larger than the intercept of the AS^{LL}. When the AD^{LL} and the AS^{LL} are parallel but their intercepts differ, then there is no intersection and thus no equilibrium with $R = 0$. When the AS^{LL} is strictly steeper than the AD^{LL}, then their intersection satisfies $\hat{h}^L > 0$, and there is no ZLB equilibrium with $\hat{h}^L \leq 0$.

The same argument goes through for the case where 2b) holds but 2a) doesn't.$\square$

Appendix D Loglinearization of the AD and the AS schedules using the L state as a reference point and formulas for multipliers

D.1 Loglinearization of the AD and the AS schedules

When computing multipliers it is sometimes convenient to loglinearize the AD and AS schedules about $\{\pi^L, h^L\}$, or in words the inflation rate and employment level in the ZLB equilibrium. We wish to emphasize at the outset that this is different from the LL solution which linearizes the equilibrium conditions at the zero inflation steady state. Let $\Delta\pi = \pi - \pi^L$, $\Delta h = \ln(h/h^L)$, $\Delta z = \ln(z/z^L)$, $\Delta\eta = \eta - \eta^L$, and $\Delta\tau_w = \tau_w - \tau_w^L$, then the loglinearized AD equation is

$$\begin{aligned}
1 &= (1-p)\beta d^L \frac{(1-\kappa^L-\eta^L)^\sigma (z^L)^\sigma (h^L)^\sigma}{(1-\eta)^\sigma z^\sigma h^\sigma}(1+\sigma(\Delta h+\Delta z) - \frac{\sigma\Delta\eta}{1-\kappa^L-\eta^L} - \frac{\sigma\gamma\pi^L\Delta\pi}{1-\kappa^L-\eta^L}) \\
&\quad + \frac{p\beta d^L}{1+\pi^L}\left(1-\frac{\Delta\pi}{1+\pi^L}\right) \\
&= \frac{p\beta d^L}{1+\pi^L}(1-\frac{\Delta\pi}{1+\pi^L}) + (1-\frac{p\beta d^L}{1+\pi^L})(1+\sigma(\Delta h+\Delta z) - \frac{\sigma\Delta\eta}{1-\kappa^L-\eta^L} - \frac{\sigma\gamma\pi^L\Delta\pi}{1-\kappa^L-\eta^L}).
\end{aligned}$$

Thus

$$\begin{aligned}
slope(AD) &= \frac{(1-\frac{p\beta d^L}{1+\pi^L})\sigma}{\frac{p\beta d^L}{(1+\pi^L)^2} + (1-\frac{p\beta d^L}{1+\pi^L})\frac{\sigma\gamma\pi^L}{1-\kappa^L-\eta^L}} \\
icept(AD) &= -\frac{(1-\frac{p\beta d^L}{1+\pi^L})\frac{\sigma}{1-\kappa^L-\eta^L}}{\frac{p\beta d^L}{(1+\pi^L)^2} + (1-\frac{p\beta d^L}{1+\pi^L})\frac{\sigma\gamma\pi^L}{1-\kappa^L-\eta^L}}[\Delta\eta - (1-\kappa^L-\eta^L)\Delta z] \\
&= slope(AD)\Delta z - slope(AD)\frac{\Delta\eta}{1-\kappa^L-\eta^L}.
\end{aligned}$$

Loglinearizing the AS equation yields:

$$\begin{aligned}
0 &= \frac{(1-\kappa^L-\eta^L)^\sigma h_L^{\sigma+\nu}}{(1-\tau_w^L)(z^L)^{1-\sigma}}\left[1+(\sigma+\nu)\Delta h - (1-\sigma)\Delta z - \frac{\sigma\Delta\eta}{1-\kappa^L-\eta^L} + \frac{\Delta\tau_w}{1-\tau_w^L} - \frac{\sigma\gamma\pi^L\Delta\pi}{1-\kappa^L-\eta^L}\right] \\
&\quad -1-(1-p\beta d^L)\frac{\gamma}{\theta}\left[\pi^L(1+\pi^L)+(1+2\pi^L)\Delta\pi\right] \\
&= -(1-p\beta d^L)\frac{\gamma}{\theta}(1+2\pi^L)\Delta\pi + \left[(1-p\beta d^L)\frac{\gamma}{\theta}\pi^L(1+\pi^L)+1\right] \\
&\quad \times\left[(\sigma+\nu)\Delta h - (1-\sigma)\Delta z - \frac{\sigma\Delta\eta}{1-\kappa^L-\eta^L} + \frac{\Delta\tau}{1-\tau_w^L} - \frac{\sigma\gamma\pi^L\Delta\pi}{1-\kappa^L-\eta^L}\right].
\end{aligned}$$

Thus

$$\begin{aligned}
slope(AS) &= \frac{\left[(1-p\beta d^L)\frac{\gamma}{\theta}\pi^L(1+\pi^L)+1\right](\sigma+\nu)}{(1-p\beta d^L)\frac{\gamma}{\theta}(1+2\pi^L) + \left[(1-p\beta d^L)\frac{\gamma}{\theta}\pi^L(1+\pi^L)+1\right]\frac{\sigma\gamma\pi^L}{1-\kappa^L-\eta^L}} \\
icept(AS) &= \frac{\left[(1-p\beta d^L)\frac{\gamma}{\theta}\pi^L(1+\pi^L)+1\right]}{(1-p\beta d^L)\frac{\gamma}{\theta}(1+2\pi^L) + \left[(1-p\beta d^L)\frac{\gamma}{\theta}\pi^L(1+\pi^L)+1\right]\frac{\sigma\gamma\pi^L}{1-\kappa^L-\eta^L}} \\
&\quad \times[-(1-\sigma)\Delta z - \frac{\sigma\Delta\eta}{1-\kappa^L-\eta^L} + \frac{\Delta\tau}{1-\tau_w^L}] \\
&= slope(AS)\frac{1}{\sigma+\nu}[-(1-\sigma)\Delta z - \frac{\sigma\Delta\eta}{1-\kappa^L-\eta^L} + \frac{\Delta\tau}{1-\tau_w^L}].
\end{aligned}$$

Loglinearizing the aggregate resource constraint $c_L = (1 - \eta^L - \kappa^L) z^L h^l$ yields

$$\Delta c = \Delta h + \Delta z - \frac{\Delta \eta}{1 - \kappa^L - \eta^L} - \frac{\gamma \pi^L \Delta \pi}{1 - \kappa^L - \eta^L}.$$

Loglinearizing GDP $gdp^L = (1 - \kappa^L) z^L h^L$ yields

$$\Delta gdp = \Delta h + \Delta z - \frac{\gamma \pi^L \Delta \pi}{1 - \kappa^L}.$$

D.2 Multiplier formulas

Our fiscal multiplier measures for the NL solution are based on the above system that has been loglinearized at $\{\pi^L, h^L\}$.

Labor tax multiplier

Note that

$$\Delta h = \frac{icept(AS) - icept(AD)}{slope(AD) - slope(AS)},$$

and

$$\Delta \pi = slope(AD) \Delta h + icept(AD) = slope(AS) \Delta h + icept(AS).$$

The labor tax multiplier on employment is thus

$$\frac{\partial \Delta h}{\partial \Delta \tau_w} = \frac{1}{slope(AD) - slope(AS)} \frac{\partial icept(AS)}{\partial \Delta \tau_w} = \frac{1}{\frac{slope(AD)}{slope(AS)} - 1} \frac{1}{\sigma + \nu} \frac{1}{1 - \tau_w}.$$

And the multiplier on inflation is:

$$\frac{\partial \Delta \pi}{\partial \Delta \tau_w} = slope(AD) \frac{\partial \Delta h}{\partial \Delta \tau_w} = \frac{slope(AD)}{\frac{slope(AD)}{slope(AS)} - 1} \frac{1}{\sigma + \nu} \frac{1}{1 - \tau_w}.$$

These multipliers are the *marginal* ones, for they are derived from elasticities.

Clearly, the slopes of the AD and the AS schedules are crucial for the multipliers. The sign of the multiplier on employment is positive when the relative slope, $slope(AD)/slope(AS)$, is bigger than one, and negative when it is less than one. Therefore, whenever the AD and the AS schedules have different signs, the multiplier on employment is negative. The multiplier is positive only when both schedules have the same signed slopes and the AD schedule is steeper. The absolute size of the multiplier explodes as the two schedules' slopes become closer.

Government purchase multiplier

To calculate the government purchases multiplier, it is convenient to start by deriving the multipliers associated with perturbations in the share of government purchases in output:

$$\frac{\partial \Delta h}{\partial \Delta \eta} = \frac{\frac{slope(AD)}{slope(AS)} - \frac{\sigma}{\sigma+\nu}}{\frac{slope(AD)}{slope(AS)} - 1} \frac{1}{1-\kappa^L-\eta^L}$$

and

$$\begin{aligned}\frac{\partial \Delta \pi}{\partial \Delta \eta} &= slope(AD)\frac{\frac{slope(AD)}{slope(AS)} - \frac{\sigma}{\sigma+\nu}}{\frac{slope(AD)}{slope(AS)} - 1} \frac{1}{1-\kappa^L-\eta^L} - slope(AD)\frac{1}{1-\kappa^L-\eta^L} \\ &= \frac{slope(AD)}{\frac{slope(AD)}{slope(AS)} - 1}\frac{\nu}{\sigma+\nu}\frac{1}{1-\kappa^L-\eta^L}\end{aligned}$$

$$\frac{\partial \Delta gdp}{\partial \Delta \eta} = \frac{\partial \Delta h}{\partial \Delta \eta} - \frac{\gamma \pi^L}{1-\kappa^L}\frac{\partial \Delta \pi}{\partial \Delta \eta}.$$

Since $\Delta g = \Delta\eta/\eta^L + \Delta h$,

$$\frac{\partial \Delta g}{\partial \Delta \eta} = \frac{1}{\eta^L} + \frac{\partial \Delta h}{\partial \Delta \eta}.$$

We can then calculate the multipliers associated with perturbations in the level of government purchases as follows

$$\begin{aligned}\text{Government purchases employment multiplier} &: \left(h^L \times \frac{\partial \Delta h}{\partial \Delta \eta}\right) / \left(g^L \times \frac{\partial \Delta g}{\partial \Delta \eta}\right) \\ \text{Government purchases GDP multiplier} &: \left(gdp^L \times \frac{\partial \Delta gdp}{\partial \Delta \eta}\right) / \left(g^L \times \frac{\partial \Delta g}{\partial \Delta \eta}\right) \\ \text{Government purchases inflation multiplier} &: \left(\frac{\partial \Delta \pi}{\partial \Delta \eta}\right) / \left(g^L \times \frac{\partial \Delta g}{\partial \Delta \eta}\right).\end{aligned}$$

Note that the government purchases increase with η when $\partial\Delta g/\partial\Delta\eta$ is positive. In such a case, the sign of the consumption response determines whether the government purchase multiplier on GDP is bigger than or less than one. Because the Euler equation implies that consumption and inflation are positively related when the nominal rate is constant, it suffices to know whether the inflation response is positive or not. What determines its sign and size is

$$slope(AD)/\{\frac{slope(AD)}{slope(AS)} - 1\}.$$

If the AD schedule is upward-sloping, the inflation response is positive when the AS schedule is also upward-sloping but flatter than the AD schedule. If both schedules are upward-sloping and the AS schedule is steeper, then the inflation response is negative. If the AD schedule is instead downward-sloping, the inflation response is positive either (i) when the AS schedule is upward-sloping, or (ii) when the AS schedule is downward-sloping and steeper than the AD schedule.

In a Sunspot equilibrium $0 < slope(AD) < slope(AS)$. Thus, the previous reasoning implies that inflation falls. It follows that employment falls in response to an increase in government purchases if

$$\frac{\sigma}{\sigma+\nu} slope(AS) < slope(AD)$$

and increases otherwise.

Effects of a technology shock

The responses of employment, output and inflation to a change in technology can be derived in the following way

$$\frac{\partial \Delta h}{\partial \Delta z} = \frac{\frac{\sigma-1}{\sigma+\nu} - \frac{slope(AD)}{slope(AS)}}{\frac{slope(AD)}{slope(AS)} - 1},$$

and it follows that

$$\frac{\partial \Delta y}{\partial \Delta z} = \frac{\partial \Delta h}{\partial \Delta z} + 1$$

and

$$\frac{\partial \Delta \pi}{\partial \Delta z} = slope(AS) \left[\frac{\partial \Delta h}{\partial \Delta z} + \frac{\sigma - 1}{\sigma + \nu} \right].$$

Observe, that output increases and employment and inflation fall in response to an improvement in technology when $\sigma = 1$ in a New ZLB equilibrium since $slope(AD) < 0 < slope(AS)$. In a Sunspot ZLB equilibrium we have $(0 < slope(AD) < slope(AS))$ and it follows that an improvement in technology increases employment, output and the inflation rate when $\sigma = 1$.

Appendix E Slope of the loglinear New Keynesian Phillips Curve

This section shows how our GR calibration targets and p, the expected duration of zero interest rates restrict the size of $slope(NKPC)$.

Denote the slope coefficient (on output) in the LL New Keynesian Phillips curve by

$$slope(NKPC) := \frac{\theta(\sigma+\nu)}{\gamma},$$

. Then the LL AS schedule can be written as

$$\pi^L = \frac{slope(NKPC)}{(1-p\beta)}\hat{h}^L + \frac{slope(NKPC)}{(1-p\beta)(\sigma+\nu)}\Big[-\sigma\frac{\hat{\eta}^L}{1-\eta} + \frac{\hat{\tau}_w^L}{1-\tau_w} - (1-\sigma)\hat{z}^L\Big]. \tag{2}$$

This relationship holds not only for our model but also for a wide range of NK models including those with Calvo price setting. Importantly, $slope(NKPC)$ is independent of $(p, \hat{d}^L)$.

First, we show that **if $\hat{z}^L = \hat{\eta}^L = \hat{\tau}^L = 0$, one is unable to reproduce the GR target with high p unless $slope(NKPC)$ is sufficiently low**. This scenario is interesting because a number of recent papers have considered specifications in which a single shock to d induces the ZLB.

Under the stated assumptions,

$$1 - p\beta = slope(NKPC)\frac{\hat{h}^L}{\pi^L} \Leftrightarrow p = \frac{1}{\beta}\left[1 - slope(NKPC)\frac{\hat{h}^L}{\pi^L}\right].$$

Our GR target is $(\hat{h}^L, \pi^L) \approx (-0.07, -0.01/4)$, hence $\hat{h}^L/\pi^L \approx 28$. This implies the following:

(A) For the model to reproduce the GR target, it is necessary that $slope(NKPC) \leq 1/28 \approx 0.036$ (this is implied by the non-negativity of p)

(B) When $\beta \approx 1$, the value of p with which the model reproduces the GR targets is reported in Table 2:

To put these numbers into perspective, consider values of the Calvo parameter implied by these values of $slope(NKPC)$. In the NK model with Calvo price setting and a homogeneous labor market, $slope(NKPC)$ is given by the formula

$$slope(NKPC) = \frac{(1-\alpha)(1-\beta\alpha)}{\alpha}(\sigma+\nu),$$

Table 2: Values of p and $slope(NKPC)$ that reproduce the GR targets

p	$slope(NKPC)$
0.16	0.03
0.44	0.02
0.58	0.015
0.66	0.012
0.72	0.01
0.86	0.005

where α is the Calvo parameter. If $(\sigma, \nu) = (1, 1)$ and $\beta \approx 1$, the right hand side equals 0.01 when the Calvo parameter α is as high as 0.93, and equals 0.02 when α is around 0.905. If $(\sigma, \nu) = (1, 0.28)$ and $\beta \approx 1$ as in our baseline specification, the right hand side equals 0.01 when α is around 0.916, and equals 0.02 when α is around 0.883.[16] Intuitively, the GR target inflation rate is so much lower than the target output decline that the New Keynesian Phillips Curve has to be very flat in order to be consistent with the target.

Note that the arguments so far are conditional on $\hat{z}^L = 0$. Deep recessions may bring about some production efficiency loss through e.g. resource misallocation, and/or lower utilization rates for production factors. When $\hat{z}^L < 0$ is allowed, then the loglinear AS schedule AS^{LL} is given by

$$\pi^L = \frac{slope(NKPC)}{(1-p\beta)}\hat{y}^L - \frac{slope(NKPC)}{(1-p\beta)}\frac{1+\nu}{\sigma+\nu}\hat{z}^L \tag{3}$$

where we have substituted out $\hat{h}^L$ using $\hat{y}^L = \hat{h}^L + \hat{z}^L$. This is because we are fixing the target values for inflation rate and GDP, and with a technology shock GDP and labor input are different. This leads to the following expression

$$p = \frac{1}{\beta}\left[1 - slope(NKPC)\left\{\frac{\hat{y}^L}{\pi^L} - \frac{1+\nu}{\sigma+\nu}\frac{\hat{z}^L}{\pi^L}\right\}\right]. \tag{4}$$

For pre-specified targets $(\hat{y}^L, \pi^L) < (0, 0)$, lowering $\hat{z}^L < 0$ increases the implied value for p for given preference parameters and $slope(NKPC)$. For instance, If $\sigma = 1$ and $\beta \approx 1$, then a value of p of 0.76 can be produced by $slope(NKPC)$ of about 0.015 together with $\hat{z}^L = -0.03$, or $slope(NKPC)$ of about 0.02 in conjunction with $\hat{z}^L = -0.04$. This is intuitive: for a given $\hat{y}^L$, negative technology shocks add inflationary pressure, and the NKPC does not need be so flat to produce a small amount of disinflation together with a relatively large decline in output.

It is worth mentioning that the above discussion does not use any information about the

[16] In our model, $slope(NKPC)$ is about 0.021 and thus it corresponds to a Calvo parameter of $\alpha \approx 0.883$.

AD schedule, and hence these results will also obtain in the true nonlinear model in situations where $slope(AS)$ and $slope(AS^{LL})$ are close.

Next, we point out that **when p is close to one an asymptote or a Conventional ZLB equilibrium is possible only if $slope(NKPC)$ is very small**. To understand this, observe that $slope(AD^{LL}) \gtreqless slope(AS^{LL})$ can be written as

$$\frac{(1-p)(1-p\beta)}{p} \gtreqless \frac{slope(NKPC)}{\sigma}.$$

When the left hand side is larger (smaller) than the right hand side, the AD^{LL} schedule is steeper (flatter) than AS^{LL}.

This relationship has several implications. First, let $\bar{p}$ be the value of p that satisfies the above with equality. When $p \to \bar{p}$, $slope(AD^{LL})/slope(AS^{LL}) \to 1$ and the denominators in the multiplier formulas go to zero as well. This results in an asymptote with very large positive or negative fiscal multipliers on each side of it. Second, since the left hand side of this inequality is decreasing in p, if we want to entertain very high p and $slope(AD^{LL}) > slope(AS^{LL})$, then the right hand side $slope(NKPC)/\sigma$ must be sufficiently low. For example, when $\beta \approx 1$ and $p = 0.9$, the left hand side is approximately 0.0111. When $\sigma = 1$, then $slope(NKPC) < 0.0111$ must hold. This restriction is not very tight for modestly large p: e.g. for $p = 0.8$ and $\beta \approx 1$, the left hand side is around 0.05, and when $\sigma = 1$, the requirement is $slope(NKPC) < 0.05$.

Appendix F Estimation

Our Bayesian estimation procedure uses the loglinearized equilibrium conditions to solve the model and derive the likelihood function. When estimating the model we ignore the ZLB and thus we are implicitly assuming that agents assigned zero ex-ante probability to the possibility of $R = 0$ during our sample period. The estimated model has a more general shock structure than the two-state Markov model presented in Section 2 of the paper. In addition, to shocks to demand d_t and technology z_t, we allow for a shock to monetary policy ϵ_t. This makes it possible to estimate the model using three observables, the output gap, inflation and the nominal interest rate.[17] The specification of the model that is estimated is given by the following equations. The aggregate demand schedule is:

$$1 = \beta d_t E_t \frac{(1+R_t)(gdp_{t+1}(1-\eta))^{-\sigma}}{(1+\pi_{t+1})(gdp_t(1-\eta))^{-\sigma}} \tag{5}$$

[17] Our measure of the output gap uses the Congressional Budget Office measure of potential GDP.

Table 3: Parameter Estimates.

Parameter	Prior			Posterior	
	distribution	mean	std. dev.	mean	90% credible interval
ν	gamma	0.5	0.25	0.37	(0.087, 0.65)
γ	normal	150	200	495.8	(275.5, 700.3)
ϕ_y	normal	0.2	0.05	0.31	(0.25, 0.38)
ϕ_π	normal	1.5	0.2	1.67	(1.49, 1.85)
ρ	beta	0.75	0.1	0.74	(0.69,0.79)
ρ_ϵ	beta	0.75	0.1	0.58	(0.47, 0.69)
ρ_z	beta	0.75	0.12	0.95	(0.92, 0.98)
ρ_d	beta	0.75	0.1	0.89	(0.85, 0.93)
σ_ϵ	inverse gamma	0.001	0.008	0.0009	(0.0008, 0.0011)
σ_z	inverse gamma	0.01	0.008	0.007	(0.0046, 0.01)
σ_d	inverse gamma	0.002	0.008	0.0021	(0.0016, 0.0025)

These estimates use U.S. quarterly data on the output gap, inflation rate and Federal Funds rate with a sample period of 1985:I-2007:IV.

and the aggregate supply schedule is:

$$\gamma\pi_t(1+\pi_t) + (1+\tau_s)(\theta-1) = \theta\frac{(gdp_t(1-\eta))^\sigma gdp_t^\nu}{(1-\tau_w)z_t^{1+\nu}(1-\kappa_t)^\nu} + \beta d_t E_t \frac{(gdp_t(1-\eta))^\sigma gdp_{t+1}(1-\kappa_t)}{(gdp_{t+1}(1-\eta))^\sigma gdp_t(1-\kappa_{t+1})}\gamma\pi_{t+1}(1+\pi_{t+1}). \tag{6}$$

The Taylor Rule is given by:

$$R_t = \rho R_{t-1} + (1-\rho)\big(\phi_\pi\pi_t + \phi_y\hat{y}_t\big) + \epsilon_t \tag{7}$$

where $\hat{y}_t$, the log output gap, is given by:

$$\hat{y}_t = \ln((\exp(gdp_t)(1-\eta))^{\sigma/(\sigma+\nu)}/(1-\tau_w)^{1/(\sigma+\nu)}). \tag{8}$$

The shocks to demand and technology are assumed to follow AR 1 rules:

$$\log d_t = \rho_d \log d_{t-1} + u_{d,t} \tag{9}$$

$$\log z_t = \rho_z \log z_{t-1} + u_{z,t} \tag{10}$$

$$\epsilon_t = \rho_\epsilon \epsilon_{t-1} + u_{\epsilon,t} \tag{11}$$

where the shocks are assumed to be Gaussian with zero means and variance-covariance matrix

$$\Omega \equiv \begin{bmatrix} \sigma_d & 0 & 0 \\ 0 & \sigma_z & 0 \\ 0 & 0 & \sigma_\epsilon \end{bmatrix}. \tag{12}$$

We estimated the model using version 4.3.3 of DYNARE. When computing the posteriors, we specified Metropolis Hastings chains of length 60,000 and used 10 parallel chains. After some experimentation we set the scale of the jumping distribution for the Metropolis-Hastings algorithm to 0.68 which produced an acceptance ratio that ranged from 0.2-0.3. The other DYNARE options for Metropolis-Hastings were set at their default values.

Priors, posterior modes, posterior means and 5 and 95 percentiles for all estimated parameters are reported in Table 3.

Appendix G Calibration of the shocks

For our main results we start by fixing $(\beta, \sigma, \nu, \theta, \gamma)$ at a particular value. Then for a given value of p, we adjust (d^L, z^L) to hit the inflation and output targets (π^L, gdp^L). The steady-state level of technology is normalized to 1, and thus the steady-state values of all prices and allocations are known. We also know that consumption in state L is: $c^L = (1 - \kappa^L - \eta^L)gdp^L/(1-\kappa^L)$, because $z^L h^L = gdp^L/(1-\kappa^L)$.

Given p, we can solve the AD equation for d^L:

$$d^L = \left[\frac{p\beta}{1+\pi^L} + (1-p)\beta\left(\frac{c^L}{c}\right)^\sigma\right]^{-1}.$$

We then solve the AS equation to find z^L:

$$\begin{aligned} \pi^L(1+\pi^L) &= \frac{1}{1-p\beta d^L}\frac{\theta}{\gamma}\left[\frac{(1-\kappa^L-\eta^L)^\sigma (h^L)^{\sigma+\nu}}{(1-\tau_w^L)(z^L)^{1-\sigma}} - 1\right] \\ &= \frac{1}{1-p\beta d^L}\frac{\theta}{\gamma}\left[\frac{(c^L)^{\sigma+\nu}}{(1-\tau_w^L)(z^L)^{1+\nu}(1-\kappa^L-\eta^L)^\nu} - 1\right]. \end{aligned}$$

Note that all variables in this second equation are known except for z^L.

When considering the specification with constant technology, we restrict $z^L = 1$ and vary d^L and θ to hit the GR targets. The preference shock d^L is calibrated first in the same way as before. This step does not require knowledge of θ. Then we use the second equation which is derived from the AS equation, to solve for θ.

When calibrating our model to the parameterization of Denes, Eggertsson, and Gilbukh (2013), we restrict $z^L = 1$ and fix θ at their estimated value of this parameter. Instead we adjust d^L and γ to satisfy the above two equations.

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